Classification and construction for symmetry protected topological phases in interacting fermion systems

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QR Wang and ZC Gu, Phys. Rev. X 8, 011055 (2018)
QR Wang and ZC Gu, Phys. Rev. X 10, 031055 (2020)
JH Zhang, S Yang, Y Qi, ZC Gu, Phys. Rev. Research 4, 033081 (2022)
Jian-Hao Zhang, Yang Qi, Zheng-Cheng Gu, arXiv:2204.13558 (2022)

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Symmetry protected topological(SPT) phase

Gapped quantum phases without symmetry breaking and long range correlation, but can not be adiabatically connected to a trivial disorder phase without phase transition.

Two basic classes of topological phases:

Intrinsic topological phases (long-range-entanglement)

- Adiabatical paths with no symmetry, e.g. fractional quantum hall effect(FQHE), gapped quantum spin liquid.

gauging

Symmetry protected topological (SPT) phases

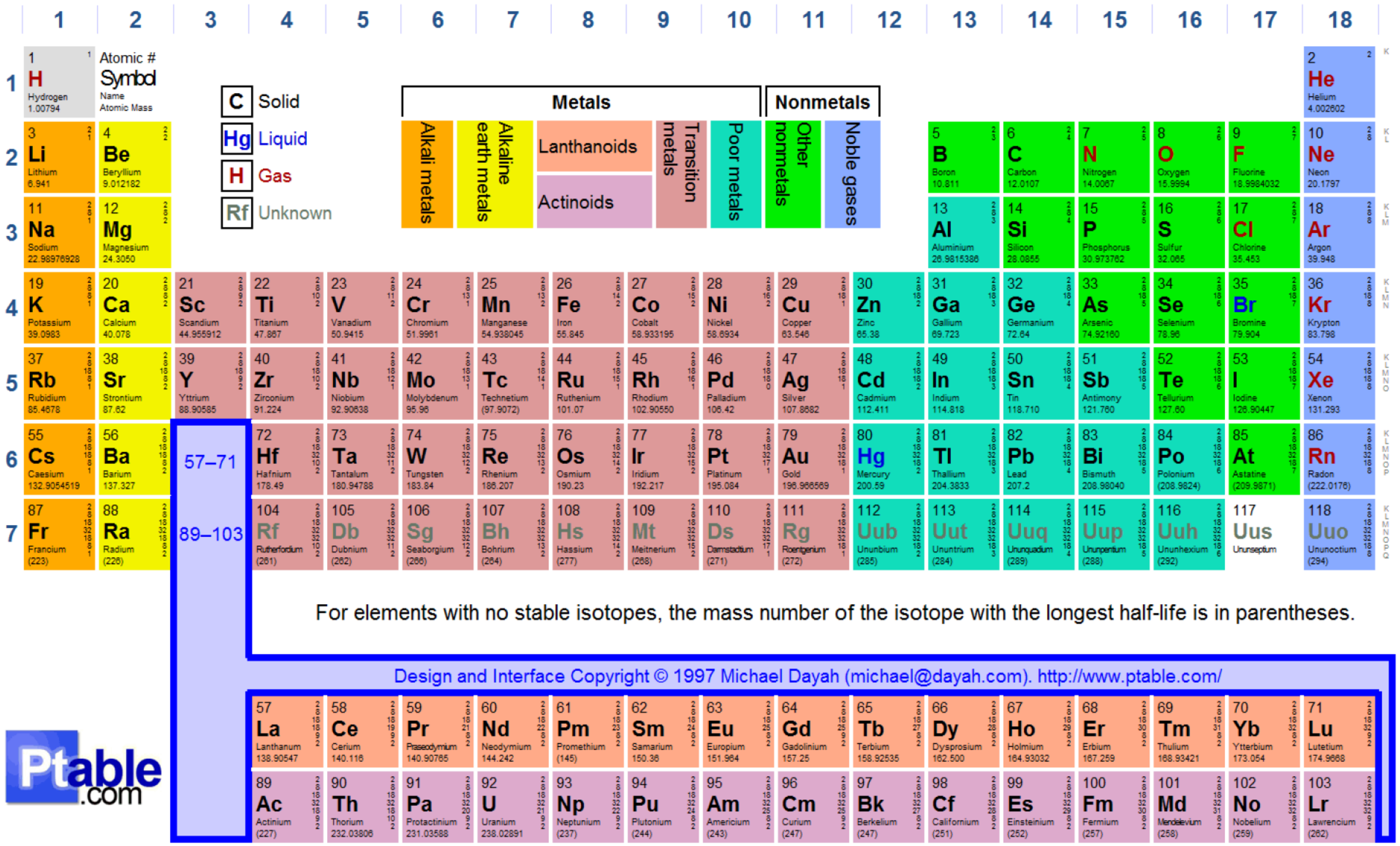
- Adiabatical paths with symmetry, e.g., topological insulator in 2D and 3D, topological superconductor in 3D

SPT phases $\xleftrightarrow{\text{symmetry breaking Hamiltonians}}$ The trivial disorder phase

(PRB 80 (15), 155131 Z C Gu, X G Wen (2009))

Why do we need a classification?

Periodic table in chemistry:



How to understand different quantum phases in many body quantum systems in a systematical way?

The answer will lead to a second quantum revolution!

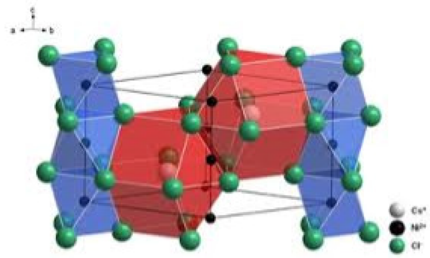
SPT phases in interacting systems

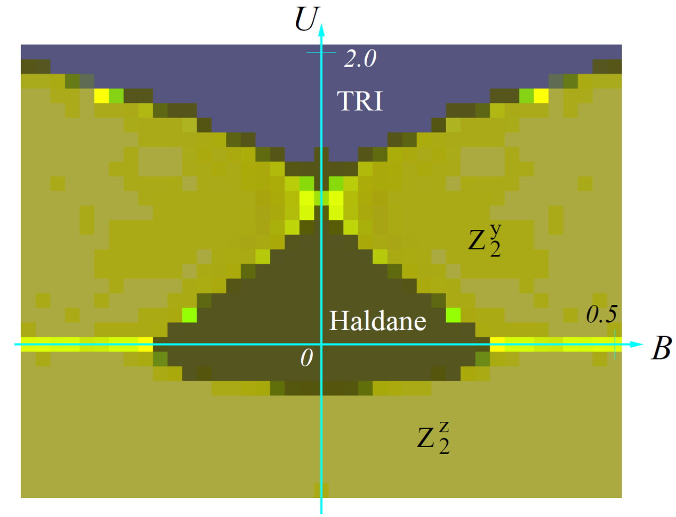
Spin one Haldane chain realizes 1D topological order

$$H = \sum_i \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2 + B S_i^x \right)$$

Uc~1(B=0)

CsNiCl3(U~B~0)

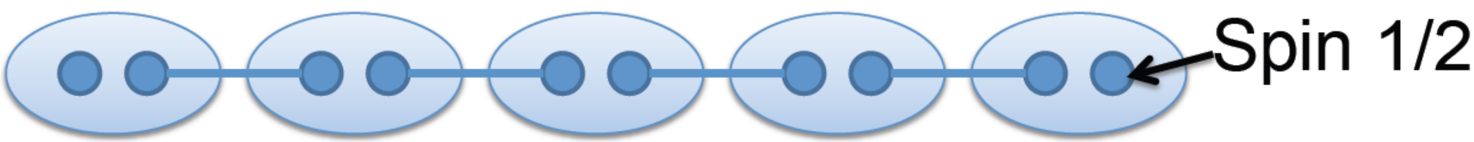




Haldane phase requires symmetry!

To define a distinct phase of matter, it is important that the characteristic properties are not destroyed by small perturbations. For the Haldane phase this was investigated in 2009 by Gu and Wen

---- Advanced information of 2016 Nobel Prize



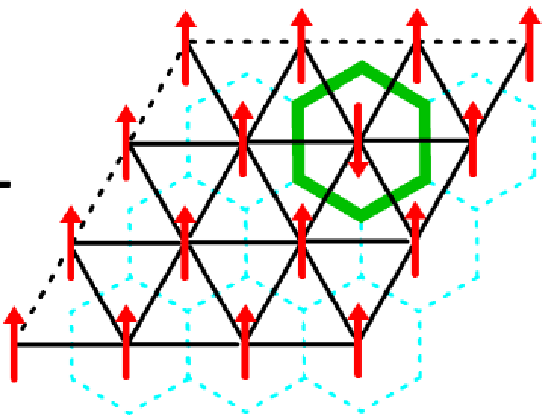
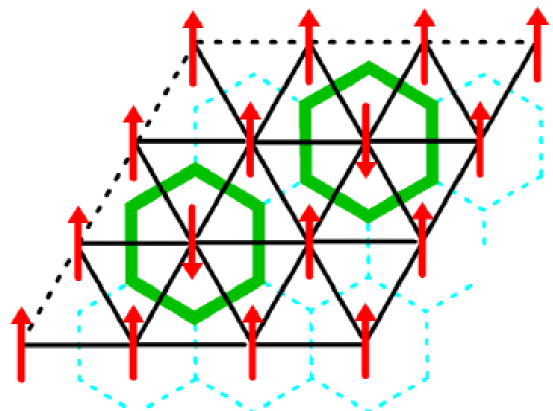
(Ian Affleck *etal.*, (1988))

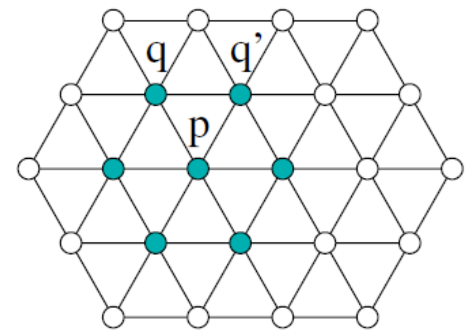
The key observation: edge states form projective representation of the symmetry group! (Xie Chen, Z C Gu, X G Wen, PRB 83, 035107 (2011))

An example of Ising SPT phase in 2D

How many different paramagnetic phases? **Two!**

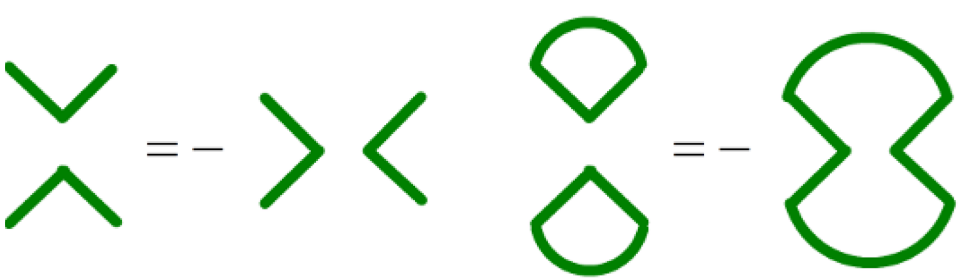
(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

$|\Psi_1\rangle = -$  $+$  $+ \dots$

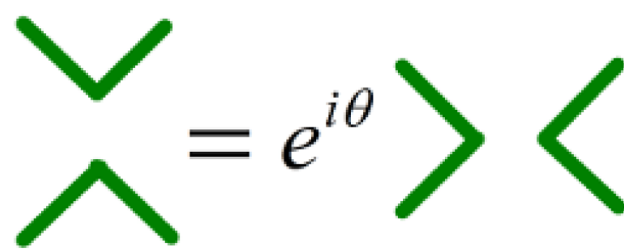


$$H_1 = -\sum_p B_p \ , \ \ B_p = -\sigma_p^x \prod_{\langle pqq'\rangle} i^{\frac{1-\sigma_q^z\sigma_{q'}^z}{2}} \qquad [B_p, B_{p'}] = 0$$

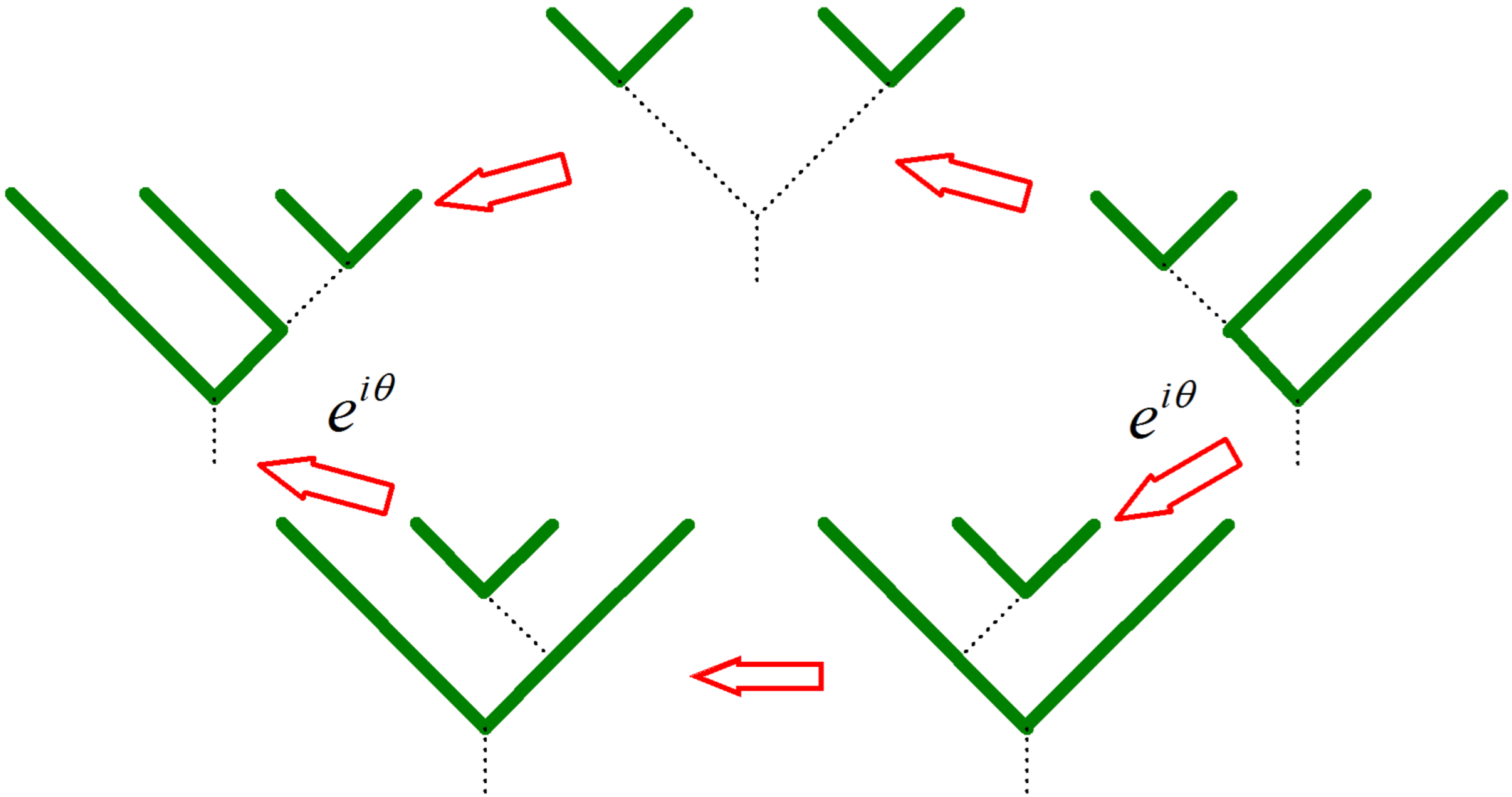
Domain deformation rule



But why not?



Topologically consistent condition for fixed point wavefunction

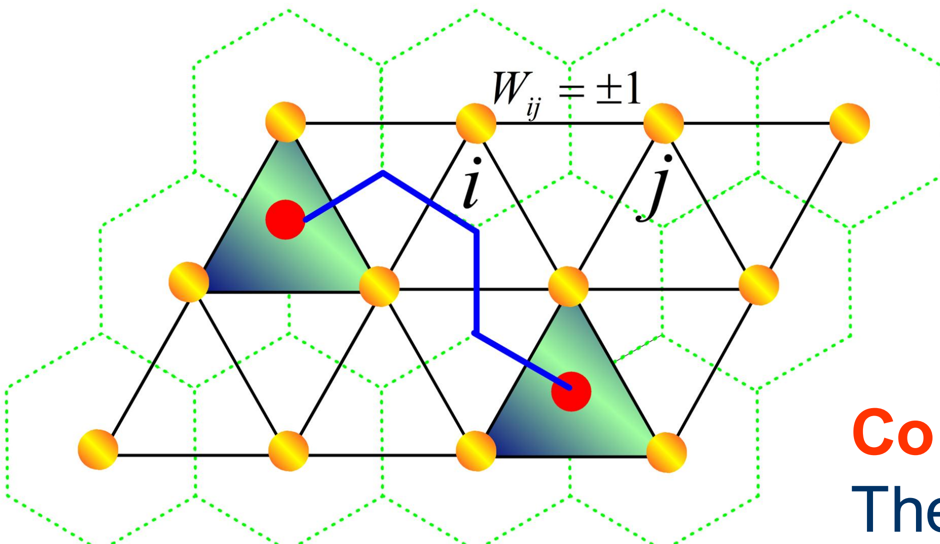


$$(e^{i\theta})^2 = 1 \Rightarrow e^{i\theta} = \pm 1$$

Bulk response and the nature of gapless edge

Assume that Ising spins carry Z_2 gauge charge and can couple to background Z_2 gauge field

Z2 gauge flux carries semion statistics!

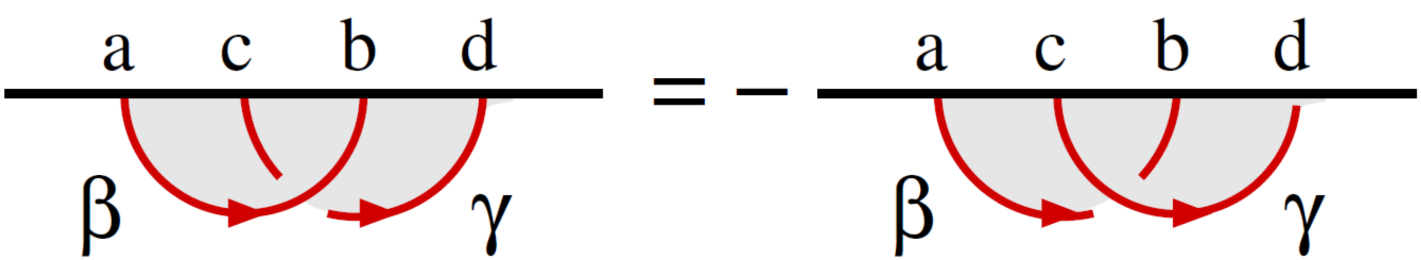


$$\widetilde{W}_\beta|0\rangle = |0\rangle$$
$$\widetilde{W}_\gamma|0\rangle = |0\rangle$$
$$\tilde{W}_\beta\tilde{W}_\gamma = -\tilde{W}_\gamma\tilde{W}_\beta$$

Contradiction

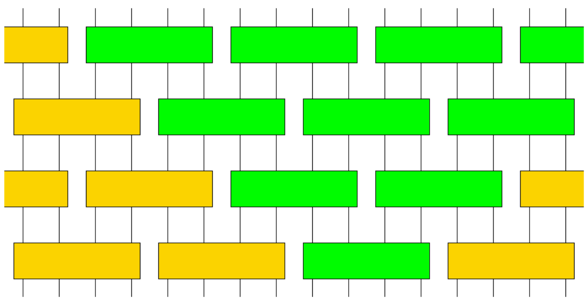
There is No 1D representation!

Non-trivial statistics of flux leads to degenerate edge states!

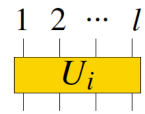


SPT phases as equivalence class of symmetric local unitary transformation

- Two states describe the same topological phase iff they are connected by finite depth local unitary(LU) transformation(acting on support space).



(a)



(b)

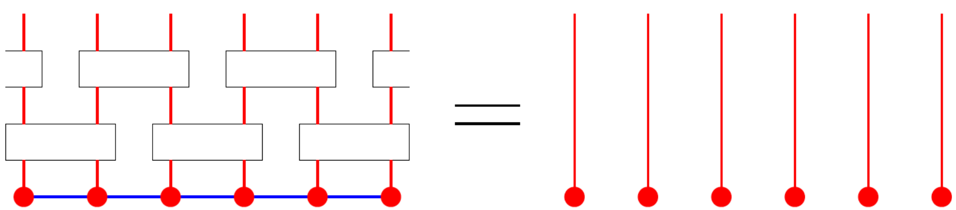
- U_i is l-local and fermion parity even(for fermion systems)

$$U_{pwl} = \prod_i U_i$$

$$|\Phi(1)\rangle \sim |\Phi(0)\rangle \text{ iff } |\Phi(1)\rangle = U^M_{circ}|\Phi(0)\rangle$$

$$U^M_{circ} = U^{(1)}_{pwl} U^{(2)}_{pwl} \cdots U^{(M)}_{pwl}$$

SPT state as short-range entangled state:

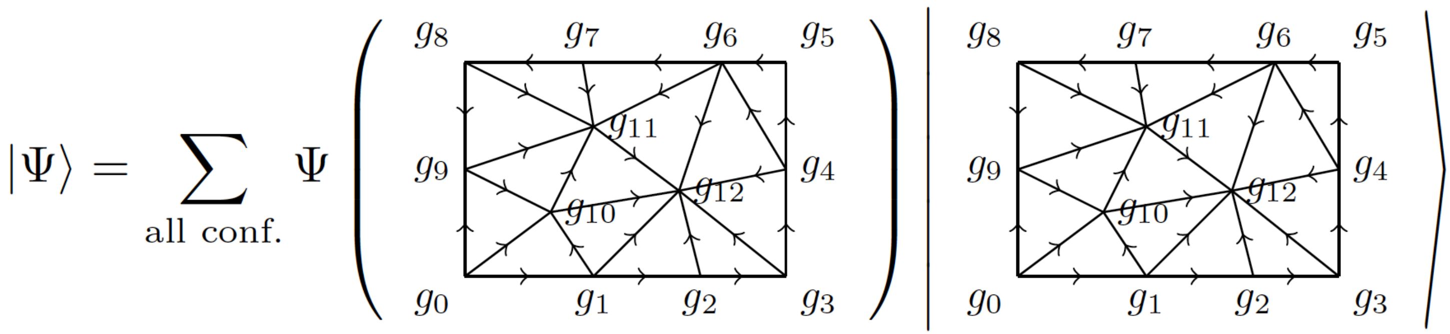


$$|\mathrm{SPT}\rangle = U^M_{circ}|\mathrm{Trivial}\rangle$$

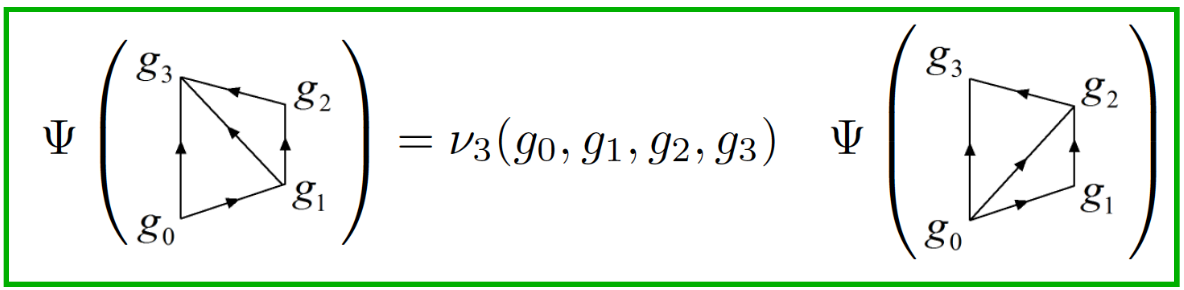
SPT phases are classified by equivalence class of symmetric LU transformation with one dimensional support space!

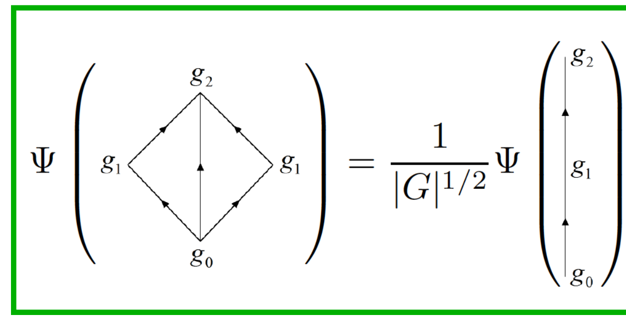
A general fixed point wavefunction for 2D SPT state

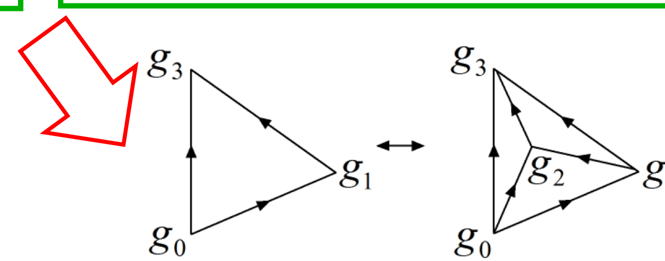
Fix point wavefunction on arbitrary triangulation



Two types of fundamental symmetric LU transformation which generates all the renormalization(retriangulation) moves for fixed point wavefunction





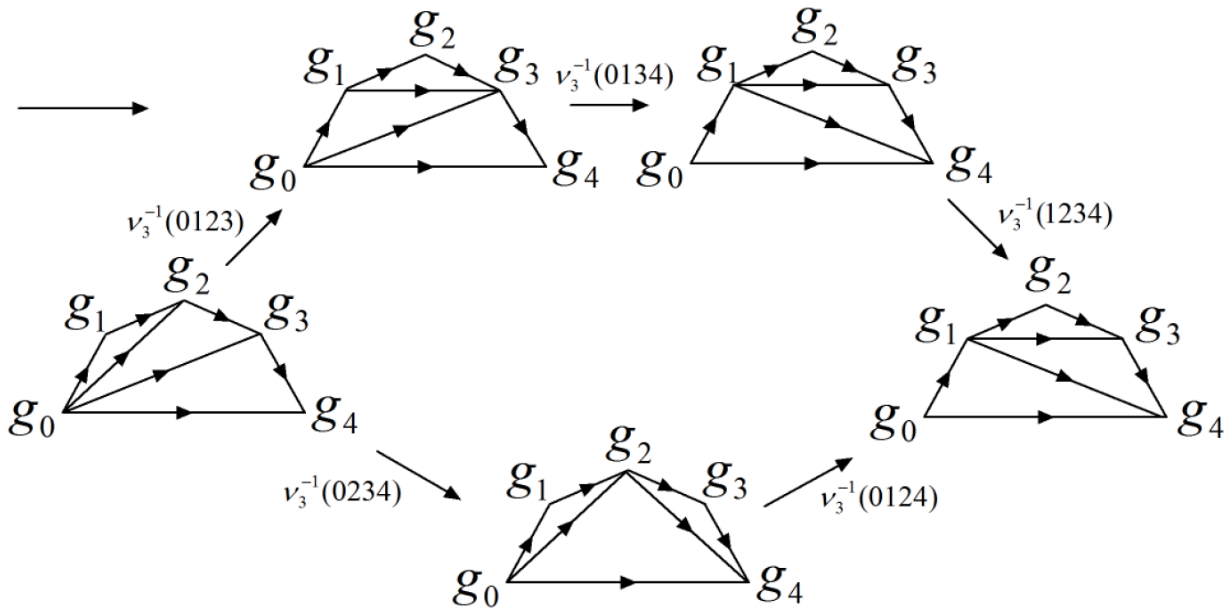


$$\nu_3(gg_0, gg_1, gg_2, gg_3) = \nu_3(g_0, g_1, g_2, g_3)$$

is a symmetric U(1) phase factor

Coherent condition and equivalent class

As a fixed point wavefucntion, different symmetric LU transformation must five rise to the same amplitude



Cocycle equation!

$$
(\mathrm{d}\nu_3)(g_0,g_1,g_2,g_3,g_4) \equiv \frac{\nu_3(g_1,g_2,g_3,g_4)\nu_3(g_0,g_1,g_3,g_4)\nu_3(g_0,g_1,g_2,g_3)}{\nu_3(g_0,g_2,g_3,g_4)\nu_3(g_0,g_1,g_2,g_4)} = 1
$$

Local basis change leads to equivalent solutions

$$
|\{g_l\}\rangle' = U_{\mu_2}|\{g_l\}\rangle = \prod_{\langle ijk\rangle} \mu_2(g_i,g_j,g_k)^{s_{\langle ijk\rangle}}|\{g_l\}\rangle
$$

$$
\nu_3'(g_0,g_1,g_2,g_3) = \nu_3(g_0,g_1,g_2,g_3)\frac{\mu_2(g_1,g_2,g_3)\mu_2(g_0,g_1,g_3)}{\mu_2(g_0,g_2,g_3)\mu_2(g_0,g_1,g_2)}
$$

Coboundary equation!

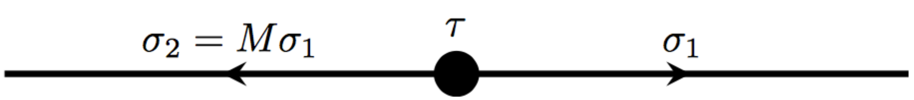
SPT phases with space group symmetry

Crystalline equivalence principle

- To compute the classification of space group SPT phases, we can just regard the space group as an internal symmetry, as long as mirror reflection symmetry is mapped into time reversal symmetry. (Ryan Thorngren and Dominic V. Else, PRX 8, 011040 (2018))

- All space group SPT phases can be constructed via the block state decoration scheme and realized as topological crystal! (Hao Song, Sheng-Jie Huang, Liang Fu, and Michael Hermele, Phys. Rev. X **7**, 011020 (2017), Zhida Song, Chen Fang, Yang Qi, Nature Communications 11, 4197 (2020))

- For example, SPT phases protected by time reversal symmetry and mirror reflection symmetry have the same classification.



For SPT protected by mirror reflection symmetry in 1D, one can just decorate a $\mathbf{Z_2}$ charge on reflection point! The two different $\mathbf{Z_2}$ eigenvalues gives rise to the correct classification!

Classifying SPT phases for interacting fermion systems protected by internal symmetry

- 1D fermionic systems can be mapped to bosonic systems with an additional unbroken fermion parity symmetry. (Xie Chen, Z C Gu, X G Wen, PRB 84, 235128 (2011))

- The braiding/three loop braiding statistics of the gauge flux/flux line is a good way to understand the 2D/3D classification. (Z.-C. Gu, M. Levin, PRB 89, 201113(R) (2014) M. Cheng, Z. Bi, Y. Z. You, Z. C. Gu, PRB 97, 205109, (2018), C Wang, CH Lin, ZC Gu, PRB 95, 195147(2017), J. R. Zhou, Q. R. Wang, C. Wang, Z. C. Gu, Nature communications 12, 1, (2021))

- Decoration of Kitaev's Majorana chains on intersection lines of symmetry domains and decoration of complex fermion on intersection points of symmetry domain walls lead to a general group super-cohomolgy theory(equivalent to Atiyah–Hirzebruch spectral sequence) for fermionic SPT phases! (Z.-C. Gu, X.-G. Wen, Phys. Rev. B 90, 115141 (2014), Q. R. Wang, ZC Gu, PRX, 8, 011055 (2018), Q. R. Wang, Z. C. Gu, PRX,10, 031055 (2020))

A revisit of Kitaev's Majorana chain

- From complex fermion to Majorana fermion:

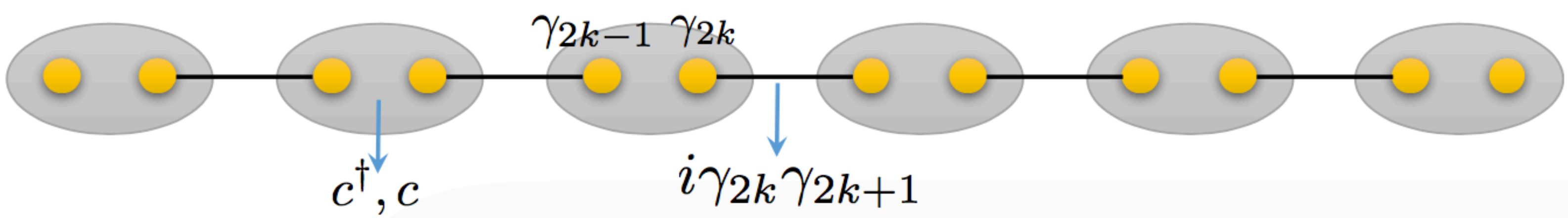
$$\begin{cases} \gamma_{2j-1} = c_j + c_j^\dagger \\ \gamma_{2j} = \frac{1}{i}(c_j - c_j^\dagger) \end{cases} \qquad \gamma_j = \gamma_j^\dagger, \quad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

- Hamiltonian and ground state:

$$H = -\sum_j (c_j^\dagger c_{j+1} + \text{h.c.}) - \sum_j (c_j c_{j+1} + \text{h.c.})$$
$$= i\sum_j \gamma_{2j}\gamma_{2j+1}$$

Majorana edge states: Two-fold degenerate ground states with different fermion parity

$$\tilde{c} = \gamma_1 + i\gamma_{2N} \qquad P_f = 1 - 2\tilde{c}^\dagger\tilde{c} = -i\gamma_1\gamma_{2N} = \pm 1$$

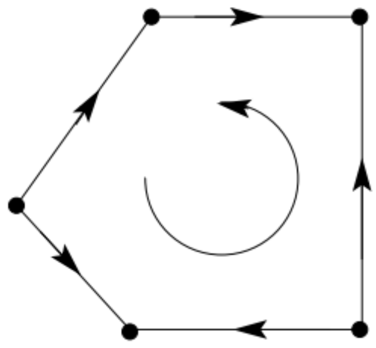


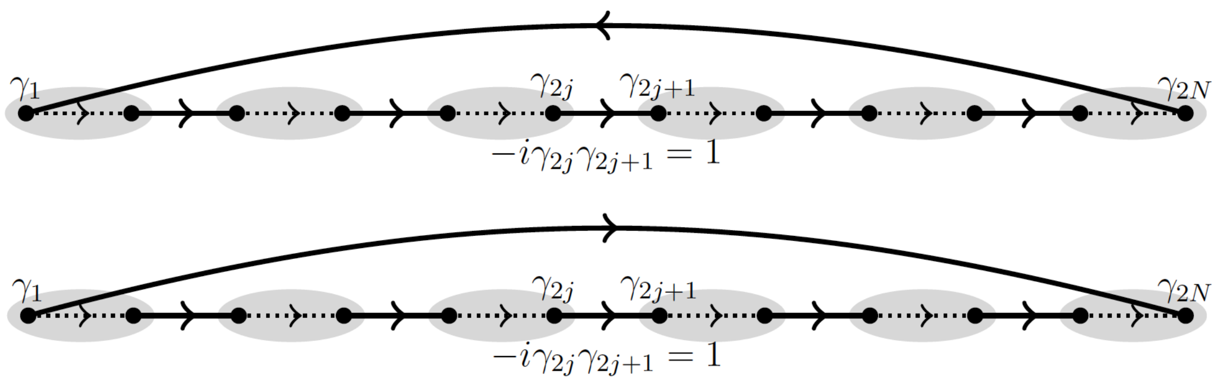
fermion parity and Kasteleyn orientation

Kasteleyn orientation:

•For a graph with edge orientation, the number of clockwise-oriented edges at every face boundary is odd

•Two Majorana dimer states have the same fermion parity, if and only if the transition graph is Kasteleyn oriented





Excitations

- Hamiltonian: $$H = \sum_j i\gamma_{2j}\gamma_{2j+1}$$
- Ground state: $$i\gamma_{2j}\gamma_{2j+1}|GS\rangle = -|GS\rangle$$
- Excited states: create a Majorana fermion at any site *k* from the ground state
$$|k\rangle = \gamma_k|GS\rangle \quad (i\gamma_{2j}\gamma_{2j+1})|k\rangle = \begin{cases} |k\rangle, & k = 2j, 2j+1 \\ -|k\rangle, & k \neq 2j, 2j+1 \end{cases}$$
- $|k\rangle$'s are not linearly independent:
- Majorana fermion hopping has phase factor $\pm i$
$$\gamma_{2j+1}|GS\rangle = i\gamma_{2j}|GS\rangle$$
- $|GS\rangle$ and $\gamma_k|GS\rangle$ have different fermion parity

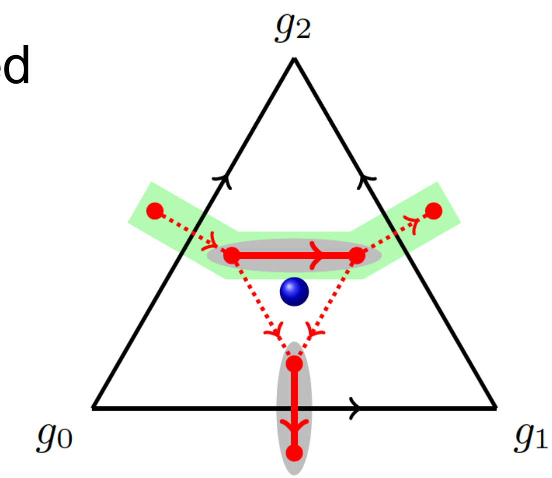
Fixed point wavefunction of 2D SPT states with total (global) symmetry $G_f = G_b \times Z_2^f$

- Hilbert space on a branched triangulation of 2D orientable manifold

$$L_{\mathcal{T}}^{2D} = \bigoplus_{f\subset F} \bigoplus_{l\subset L} \Big(\prod_{(ijk)\in f} c_{(ijk)}^{\dagger} |0\rangle \bigotimes \prod_{(ij)\in l} a_{(ij)}^{\dagger} |\tilde{0}\rangle \bigotimes \prod_{v\in V(\mathcal{T})} \mathbb{C}^{|G_b|} \Big).$$

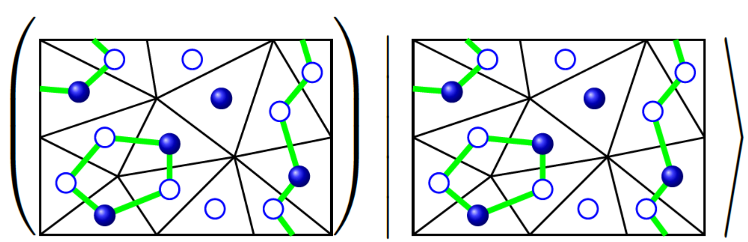
- The complex fermion a_{ij} on each edge will be splitted into a pair of Majorana fermions determined by discrete spin structure(Kasteleyn orientation).

- The complex fermion c_{ijk} is will be decorated on the intersection point of G_b symmetry domain walls. A Majorana chain formed by a_{ij} will be decorated on G_b symmetry domain wall.



- Fixed-point state is a superposition of those basis states on all possible triangulations.

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi$$

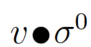


2D Discrete spin structure and Kasteleyn orientation

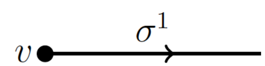
Lattice realization of spin structure:

$(n-2)$-th Stiefel-Whitney homology class $[w_{n-2}]$ which is the Poincaré dual

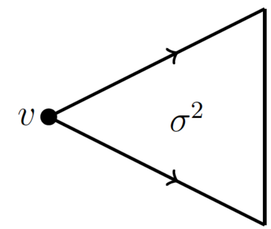
Regular pair:



(a) $\sigma^0 = v$ is a 0-simplex.

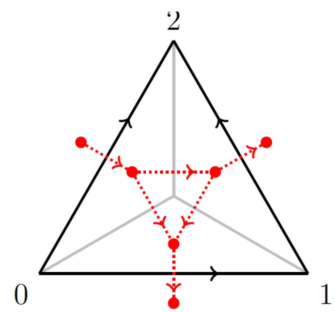


(b) σ^1 is an 1-simplex.

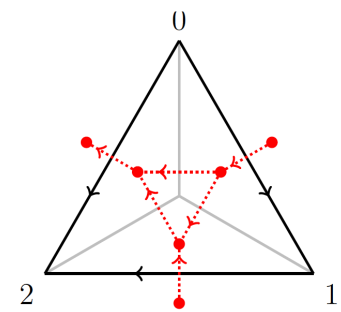


(c) σ^2 is a 2-simplex.

A convention on the resolved dual graph:



(a) Positive oriented triangle

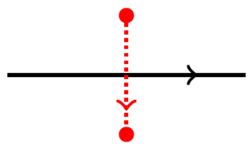


(b) Negative oriented triangle

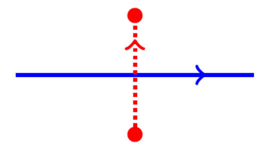
$$
\begin{aligned}
w_0 &= \sum_{v\in\mathcal{T}} \left(1 + \#\{\sigma^1 | v \subseteq \sigma^1 \text{ is regular}\} + \#\{\sigma^2 | v \subseteq \sigma^2 \text{ is regular}\}\right) \cdot v \\
&= \sum_{v\in\mathcal{T}} v \ (v \text{ is non-Kasteleyn-oriented}).
\end{aligned}
$$

A simple example:

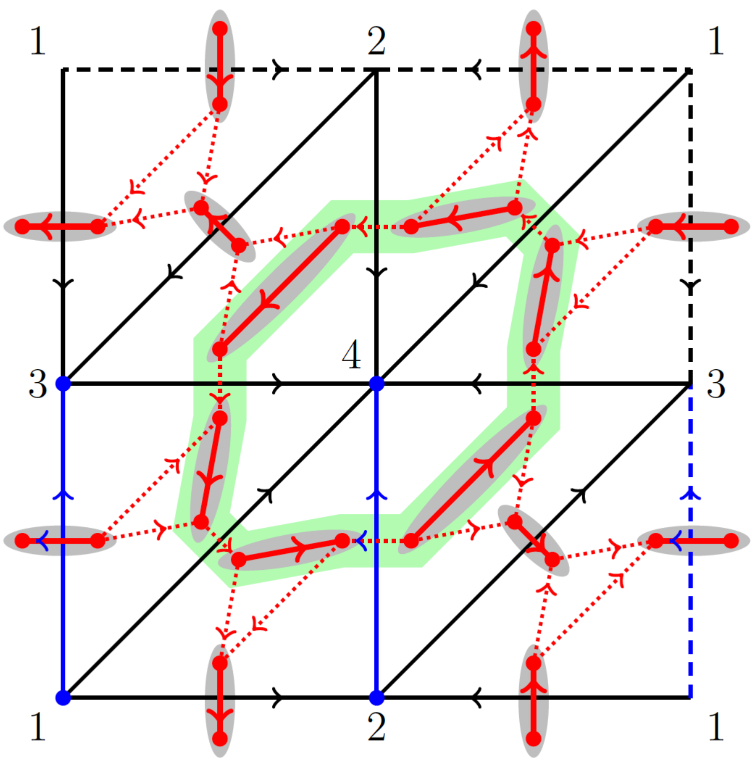
Reverse the error when crossing a singular line:



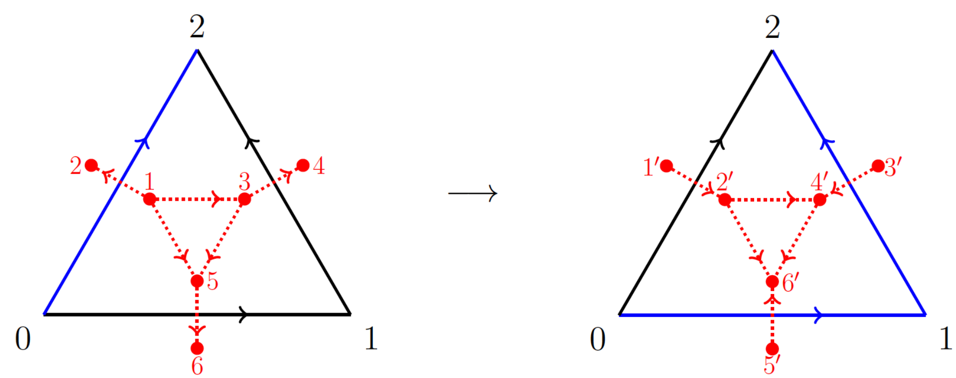
(a) Orientation convention for (red) link dual to (black) link $l \notin S$.



(b) Orientation convention for (red) link dual to (blue) singular link $l \in S$.

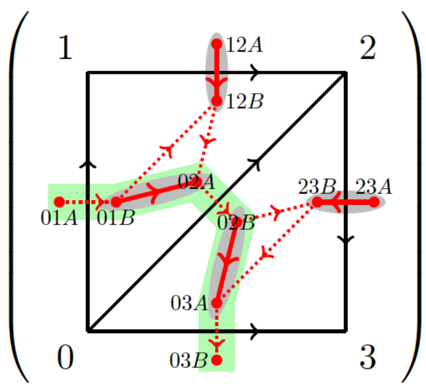
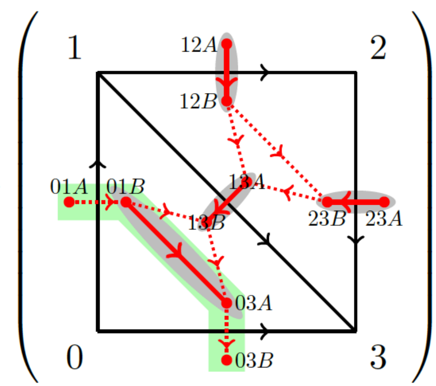


A gauge choice:



Wavefunction renormalization with fermionic symmetric local unitary transformation

- Equivalent class of fermionic symmetric local unitary transformation with support dimension one will give rise to a definition and classification of SPT phases in interacting fermion systems.

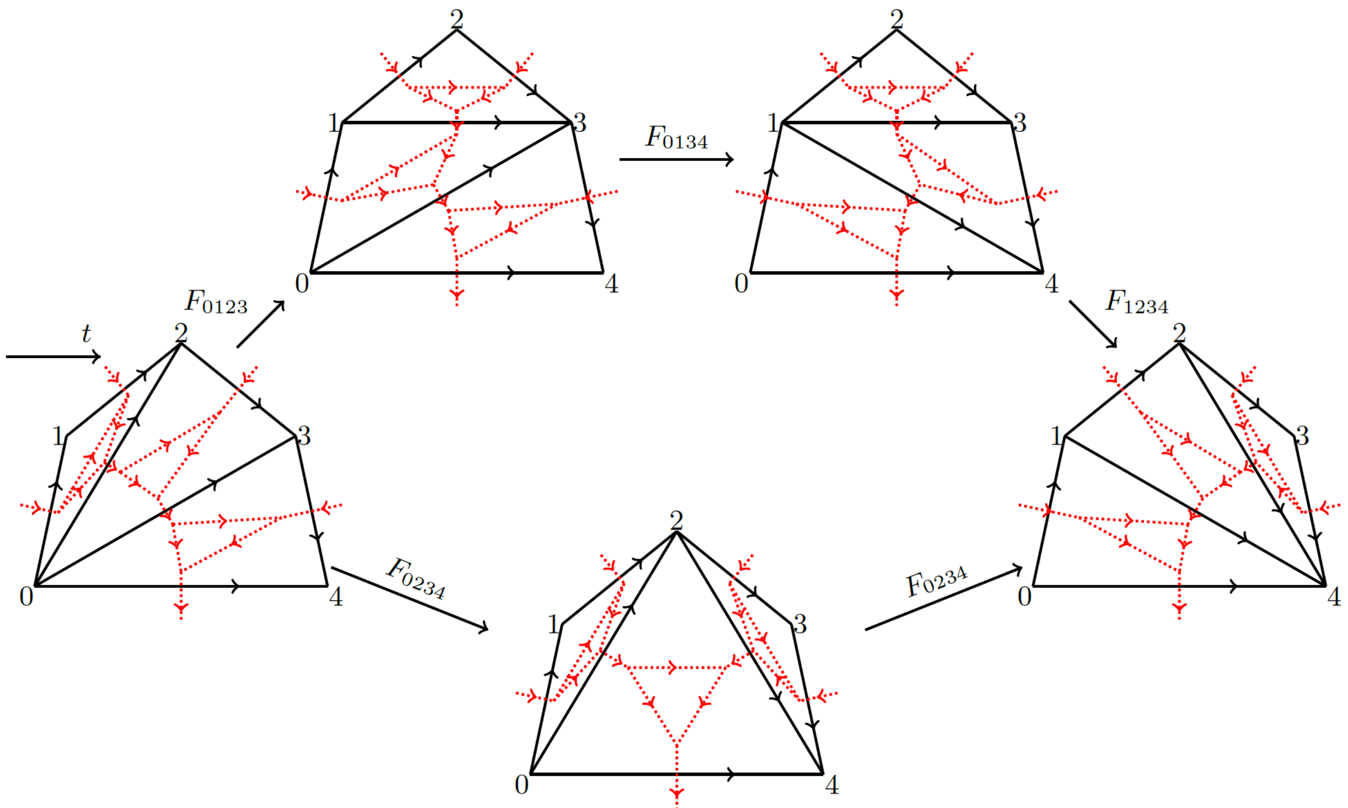
Ψ  $= F(g_0,g_1,g_2,g_3)\ \Psi$ 

$$F(g_0,g_1,g_2,g_3) = \nu_3(g_0,g_1,g_2,g_3)\, c_{(012)}^{\dagger n_2(g_0,g_1,g_2)}\, c_{(023)}^{\dagger n_2(g_0,g_2,g_3)}\, c_{(013)}^{n_2(g_0,g_1,g_3)}\, c_{(123)}^{n_2(g_1,g_2,g_3)}\, X[\tilde{n}_1(g_i,g_j)]$$

$$X[\tilde{n}_1] = 2^{1/2}\,(P_{01B,02A}P_{02B,03A})\,P_{13A,13B} \qquad P_{a,b} = (1 - i\gamma_a\gamma_b)/2$$

- In-equivalent decoration patterns of complex fermion on the intersection points of G_b symmetry domain is classified by $n_2 \in H^2(G_b, \mathbb{Z}_2)$
- In-equivalent decoration patterns of Majorana chain on the G_b symmetry domain is classified by $\tilde{n}_1 \in H^1(G_b, \mathbb{Z}_2)$

Fixed point conditions and obstruction



$$(\mathrm{d}\nu_3)(g_0,g_1,g_2,g_3,g_4) = (-1)^{Sq^2(n_2)(g_0,g_1,g_2,g_3,g_4)} = (-1)^{n_2(g_0,g_1,g_2)n_2(g_2,g_3,g_4)}$$

Obstruction free subgroup: $BH^2(G_b,\mathbb{Z}_2)$

$n_2 \in H^2(G_b,\mathbb{Z}_2)$ that satisfy $Sq^2(n_2) = 0$ in $H^4(G_b,U_T(1))$

Classification of 2D SPT phases with symmetry $G_f=G_b\times Z_2^f$ for interacting fermions: $H^1(G_b,\mathbb{Z}_2)$, $BH^2(G_b,\mathbb{Z}_2)$ and $H^3(G_b,U_T(1))$

Generic symmetry G^f for fermion systems

- A generic G^f is defined by the short exact sequence:

$$1 \to \mathbb{Z}_2^f \to G_f \to G_b \to 1$$

- It is called central extension and specified by a **Z_2** coefficient cocycle:

$$\boxed{\omega_2 \in H^2(G_b, \mathbb{Z}_2 = \{0,1\})}$$

- For a given group element in the total group G_f:

$$g_f = (P_f^{n(g)}, g_b) \in \mathbb{Z}_2^f \times G_b,$$

$$g_f \cdot h_f = \left(P_f^{n(g)}, g_b\right) \cdot \left(P_f^{n(h)}, h_b\right) := \left(P_f^{n(g)+n(h)+\omega_2(g_b,h_b)}, g_b h_b\right)$$

$$P_f^{n(g)+n(h)+\omega_2(g_b,h_b)} \in \mathbb{Z}_2^f \text{ and } g_b h_b \in G_b.$$

- The associativity of group multiplication holds naturally due to the fact that:

$$\omega_2(h,k) + \omega_2(gh,k) + \omega_2(g,hk) + \omega_2(g,h) = 0$$

Classification of general FSPT phases in 2D

$$\begin{cases} n_1 \in H^1(G_b, \mathbb{Z}_2), \\ n_2 \in C^2(G_b, \mathbb{Z}_2)/B^2(G_b, \mathbb{Z}_2)/\Gamma^2, \\ \nu_3 \in C^3(G_b, U(1)_T)/B^3(G_b, U(1)_T)/\Gamma^3. \end{cases}$$

$$\begin{cases} n_1(gg_0, gg_1) = n_1(g_0, g_1) = n_1(g_0^{-1}g_1), \\ n_2(gg_0, gg_1, gg_2) = n_2(g_0, g_1, g_2) = n_2(g_0^{-1}g_1, g_1^{-1}g_2), \\ \nu_3(g, ga, gab, gabc) = {}^{g}\nu_3(a,b,c) = \nu_3(a,b,c)^{1-2s_1(g)} \cdot \mathcal{O}_4^{\text{symm}}(g, ga, gab, gabc) \end{cases}$$

$$\begin{cases} \mathrm{d}n_1 = 0, \\ \mathrm{d}n_2 = \omega_2 \smile n_1 + s_1 \smile n_1 \smile n_1, \\ \mathrm{d}\nu_3 = \mathcal{O}_4[n_2] \end{cases}$$

$$\begin{cases} \Gamma^2 = \{\omega_2 \in H^2(G_b, \mathbb{Z}_2)\}, \\ \Gamma^3 = \{(-1)^{\omega_2 \smile n_1} \in H^3(G_b, U(1)_T) | n_1 \in H^1(G_b, \mathbb{Z}_2)\}. \end{cases}$$

$$\begin{aligned} \mathcal{O}_4^{\text{symm}}(g_0, g_1, g_2, g_3) &= (-1)^{\omega_2(g_0, g_0^{-1}g_1) n_2(123) + [s_1(g_0) + \omega_2(g_0, g_0^{-1}g_2)] \mathrm{d}n_2(0123)} \\ &= (-1)^{(\omega_2 \smile n_2 + s_1 \smile \mathrm{d}n_2)(g_0, g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3) + \omega_2(g_0, g_0^{-1}g_2) \mathrm{d}n_2(g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3)} \end{aligned}$$

$$\mathcal{O}_4[n_2](01234) = (-1)^{(\omega_2 \smile n_2 + n_2 \smile n_2 + n_2 \smile_1 \mathrm{d}n_2)(01234) + \omega_2(013)\mathrm{d}n_2(1234) + \mathrm{d}n_2(0124)\mathrm{d}n_2(0234)} (-i)^{\mathrm{d}n_2(0123)[1 - \mathrm{d}n_2(0124)]}$$

$s_1 \in H^1(G_b, \mathbb{Z}_2)$ is related to time reversal symmetry.

2-cocycle $\omega_2 \in H^2(G_b, \mathbb{Z}_2)$ which tells us how G_b is extended

Q. R. Wang, Z. C. Gu, PRX,10, 031055 (2020)

Classification of space group SPT phases for 2D interacting fermion systems

Fermionic crystalline equivalence principle

- A mirror reflection symmetry action should still be mapped onto a time reversal symmetry action. In addition, spinless(spin-1/2) fermion systems should be mapped onto spin-1/2(spinless) fermion system.

Bosonic Crystalline SPT phases

SG	MC	CF	B	Total
p1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
p2	$3\mathbb{Z}_2$	$4\mathbb{Z}_2$	$4\mathbb{Z}_2$	2048
p1m1	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	32
p1g1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
c1m1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
p2mm	0	0	$8\mathbb{Z}_2$	256
p2mg	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$3\mathbb{Z}_2$	256
p2gg	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	64
c2mm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$5\mathbb{Z}_2$	128
p4	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_4$	1024
p4mm	0	0	$6\mathbb{Z}_2$	64
p4gm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_4$	64
p3	0	$\mathbb{Z}_2$	$3\mathbb{Z}_3$	54
p3m1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	4
p31m	0	$\mathbb{Z}_2$	$\mathbb{Z}_6$	12
p6	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_6$	288
p6mm	0	0	$4\mathbb{Z}_2$	16

Spin-1/2

SG	MC	CF	B	Total
p1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
p2	0	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	16
p1m1	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	64
p1g1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
c1m1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	16
p2mm	0	$4\mathbb{Z}_2$	$4\mathbb{Z}_2$	256
p2mg	$\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2$	32
p2gg	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
c2mm	0	$3\mathbb{Z}_2$	$2\mathbb{Z}_2$	32
p4	0	$2\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	32
p4mm	0	$4\mathbb{Z}_2$	$3\mathbb{Z}_2$	128
p4gm	0	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	16
p3	0	$\mathbb{Z}_2$	$3\mathbb{Z}_3$	54
p3m1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
p31m	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	24
p6	0	$\mathbb{Z}_2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	36
p6mm	0	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	16

Spinless

(Y Ouyang, QR Wang, ZC Gu, Y Qi, arXiv:2005.06572 (2020))

Real space construction for space group FSPT phases in 2D

Three major steps:

- Cell decomposition

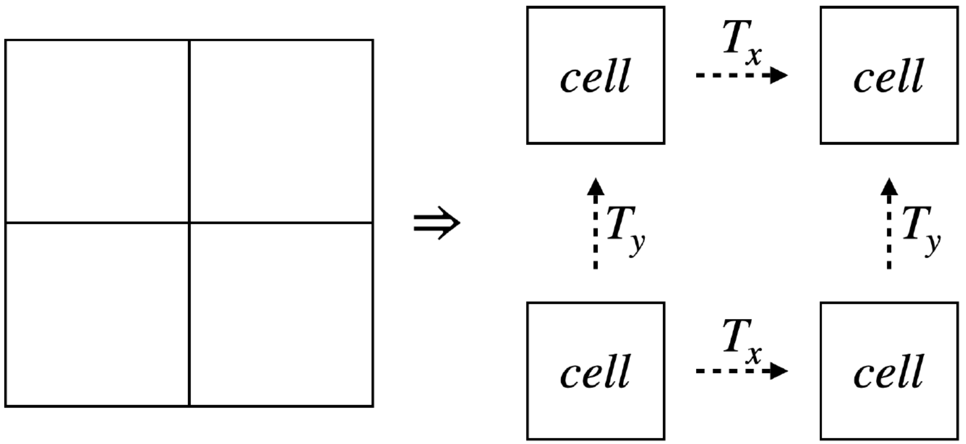
- 1D and 0D Block-state decoration(with or without symmetries): Possible obstructions for block-states; all obstruction-free block-states form a group {OFBS}

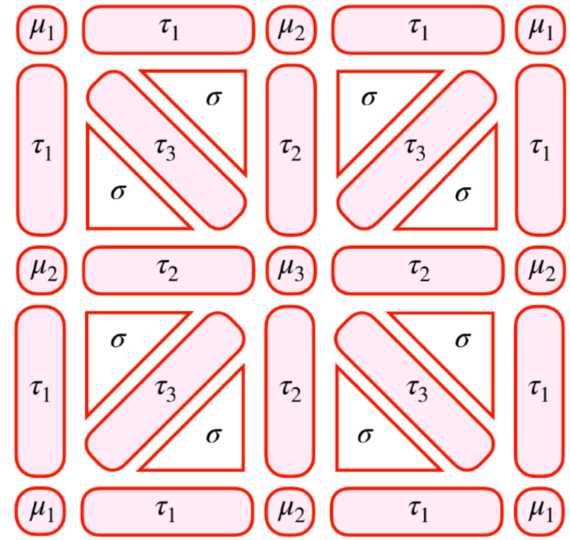
- Bubble equivalence: Some block-states might be equivalent via bubble equivalences; all trivial block-states form a group {TBS}

Topological distinct phases can be labeled by different group elements of the quotient group G= {OFBS}/{TBS}

p4m space group with spinless fermion

- Space group is composed by point group and translational symmetry.





- For wallpaper group p4m, the corresponding point group is D_4. Cell decomposition of p4m leads to an assembly of following lower dimensional blocks with corresponding on-site symmetries:

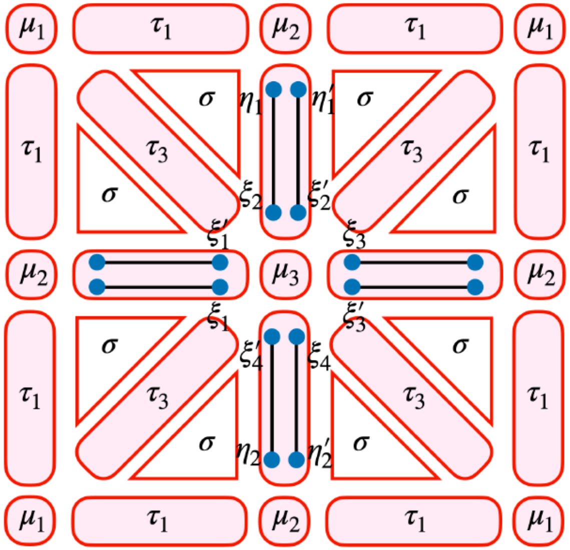
2D blocks: no on-site symmetry;

1D blocks: Z_2 on-site symmetry;

0D blocks 1 and 3: D_4 on-site symmetry; 2: D_2 on-site symmetry.

Block state decoration

- Similar as the D_4 point group case, Majorana chain decoration is not allowed. Only possible for 1D FSPT state decoration with a total symmetry $Z_2 \times Z_2^f$.(double Majorana chain)



There are two independent 1D blocks comparable with 1D FSPT decoration

$$\{\mathrm{OFBS}\}^{1\mathrm{D}}_{p4m,0} = \mathbb{Z}_2^2$$

- Similar as the D_4/D_2 point group case, 0D block-states decoration is classified by:

$$\mathcal{H}^1\left[\mathbb{Z}_2^f \times (\mathbb{Z}_n \rtimes \mathbb{Z}_2), U(1)\right] = \mathbb{Z}_2^3$$

There are three independent 0D blocks, each can be labelled as fermion parity and two mirror eigenvalues

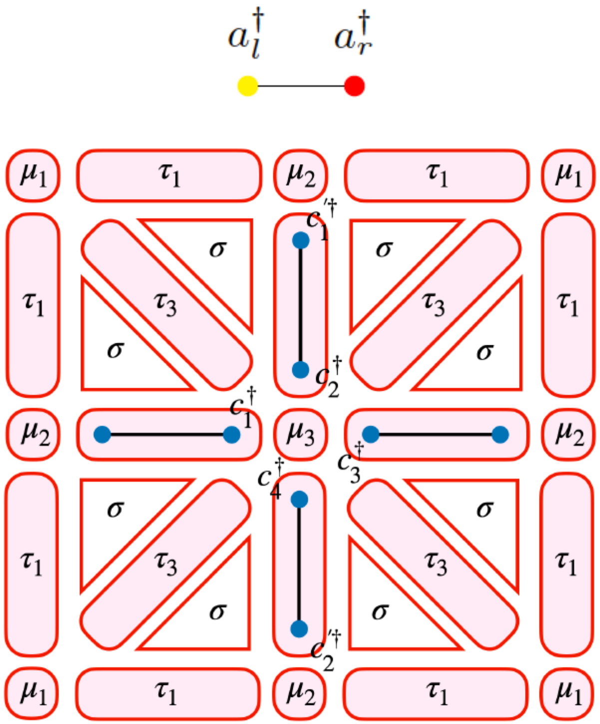
$$\{\mathrm{OFBS}\}^{0\mathrm{D}}_{p4m,0} = \mathbb{Z}_2^9 \qquad [(\pm,\pm,\pm),(\pm,\pm,\pm),(\pm,\pm,\pm)]$$

Final classification of obstruction free block states:

$$\{\mathrm{OFBS}\}_{p4m,0} = \{\mathrm{OFBS}\}^{1\mathrm{D}}_{p4m,0} \times \{\mathrm{OFBS}\}^{0\mathrm{D}}_{p4m,0} = \mathbb{Z}_2^2 \times \mathbb{Z}_2^9 = \mathbb{Z}_2^{11}$$

Bubble equivalence

1D bubble construction can be realized as creating a pair of complex fermion on all 1D blocks.:



$$|\psi\rangle^{\mu_3}_{p4m} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger|0\rangle,\ \ |\psi\rangle^{\mu_2}_{p4m} = c_1'^\dagger c_2'^\dagger|0\rangle$$

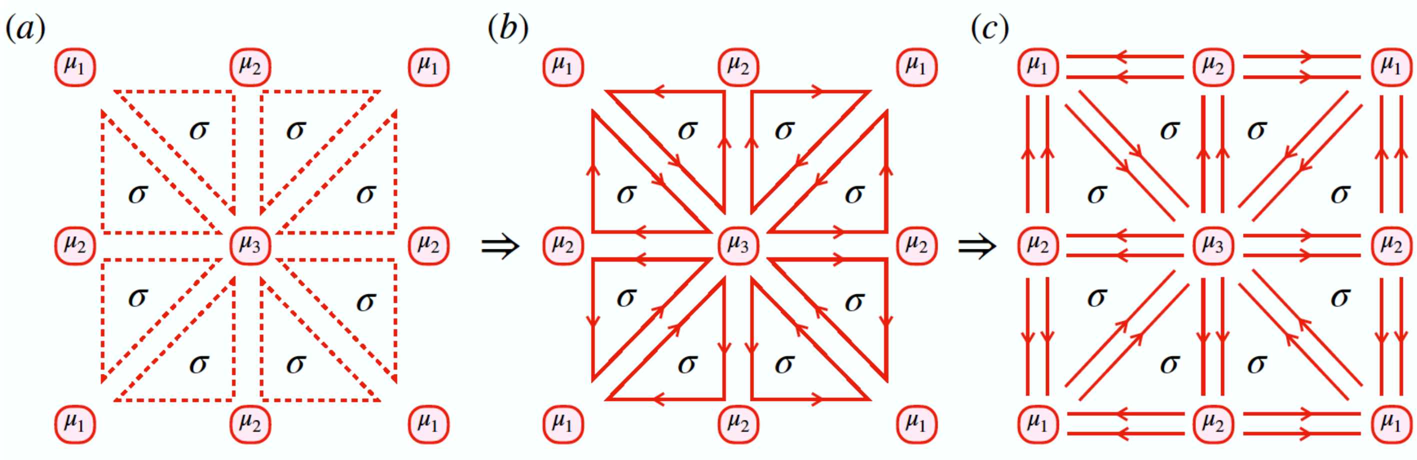
$$\boldsymbol{M}_{\tau_3}|\psi\rangle^{\mu_3}_{p4m} = c_2^\dagger c_1^\dagger c_4^\dagger c_3^\dagger|0\rangle = |\psi\rangle^{\mu_3}_{p4m}$$

$$\boldsymbol{M}_{\tau_2}|\psi\rangle^{\mu_3}_{p4m} = c_1^\dagger c_4^\dagger c_3^\dagger c_2^\dagger|0\rangle = -|\psi\rangle^{\mu_3}_{p4m}$$

$$\boldsymbol{M}_{\tau_1}|\psi\rangle^{\mu_2}_{p4m} = c_2'^\dagger c_1'^\dagger|0\rangle = -|\psi\rangle^{\mu_2}_{p4m}$$

$$\boldsymbol{M}_{\tau_2}|\psi\rangle^{\mu_2}_{p4m} = c_1'^\dagger c_2'^\dagger|0\rangle = |\psi\rangle^{\mu_2}_{p4m}$$

2D bubble construction can be realized as creating a Majorana chain on the boundaries of all 2D blocks.



Classification results

- 1D bubble constructions can be characterized by three integers(mod 2), describing the changing of two mirror eigenvalues for each 0D block

$$[(+,(-1)^{n_1},(-1)^{n_3}),(+,(-1)^{n_2},(-1)^{n_1}),(+,(-1)^{n_2},(-1)^{n_3})]$$

$$\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{0D}}=\mathbb{Z}_2^3$$

$$G_{p4m,0}^{\mathrm{0D}}=\{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{0D}}/\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{0D}}=\mathbb{Z}_2^9/\mathbb{Z}_2^3=\color{red}{\mathbb{Z}_2^3}\times\color{blue}{\mathbb{Z}_2^3}$$

- 2D bubble constructions will trivialize one of the 1D FSPT decoration

$$\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{1D}}=\mathbb{Z}_2$$

$$G_{p4m,0}^{\mathrm{1D}}=\{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{1D}}/\{\mathrm{TBS}\}_{p4m,0}^{\mathrm{1D}}=\mathbb{Z}_2^2/\mathbb{Z}_2=\color{red}{\mathbb{Z}_2}$$

$$\mathcal{G}_{p4m,0}=\{\mathrm{OFBS}\}_{p4m,0}/\{\mathrm{TBS}\}_{p4m,0}=\mathbb{Z}_2^{11}/\mathbb{Z}_2^4=\color{red}{\mathbb{Z}_2^4}\times\color{blue}{\mathbb{Z}_2^3}$$

- In general, the stacking of 1D block state might lead to a nontrivial 0D block state, which implies nontrivial group structure.

Classification of TSC in 2D spin-1/2/spinless fermion systems

G_b	$E^{1D}_{1/2}$	$E^{0D}_{1/2}$	$\mathcal{G}_{1/2}$
$p1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
$p2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4^4$	$\mathbb{Z}_4 \times \mathbb{Z}_8^3$
pm	$\mathbb{Z}_2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_4 \times \mathbb{Z}_8$
pg	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
cm	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
pmm	0	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8$
pmg	$\mathbb{Z}_2^2$	$\mathbb{Z}_4^3$	$\mathbb{Z}_4 \times \mathbb{Z}_8^2$
pgg	$\mathbb{Z}_2^2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
cmm	$\mathbb{Z}_2$	$\mathbb{Z}_4 \times \mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_2^4$
$p4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8^2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_8^3$
$p4m$	0	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$
$p4g$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_2^2$
$p3$	0	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$
$p3m1$	0	$\mathbb{Z}_4$	$\mathbb{Z}_4$
$p31m$	0	$\mathbb{Z}_4 \times \mathbb{Z}_3$	$\mathbb{Z}_4 \times \mathbb{Z}_3$
$p6$	$\mathbb{Z}_2$	$\mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_3$	$\mathbb{Z}_{12} \times \mathbb{Z}_8 \times \mathbb{Z}_3$
$p6m$	0	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$

G_b	E^{1D}_{0}	E^{0D}_{0}	$\mathcal{G}_{0}$
$p1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$p2$	0	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pm	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^5 \times \mathbb{Z}_2$
pg	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
cm	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pmm	0	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$
pmg	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
pgg	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
cmm	0	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
$p4$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$p4m$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^3$
$p4g$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$
$p3$	0	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
$p31m$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$
$p6$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$
$p6m$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$

- Fermionic/bosonic root phases are denoted with red/blue color.
- The 1D Majorana chain decoration, 0D complex fermion decoration and bosonic SPT phases has a one to one correspondence with internal symmetry.

(JH Zhang, S Yang, Y Qi, ZC Gu, Phys. Rev. Research 4, 033081 (2022))

Generalize into topological insulators(TI) in 2D interacting fermion systems

G_b	spinless	spin-1/2
$p1$	$\mathbb{Z}$	$\mathbb{Z}$
$p2$	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2$
pm	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$
pg	$\mathbb{Z}$	$\mathbb{Z}$
cm	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$
pmm	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2^4$	$2\mathbb{Z}\times\mathbb{Z}_2^8$
pmg	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2$
pgg	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$
cmm	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2^2$	$2\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2^4$
$p4$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$
$p4m$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2^3$	$2\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2^6$
$p4g$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_2^2$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_2^2$
$p3$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_3^3$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_2$
$p31m$	$\mathbb{Z}\times\mathbb{Z}_3\times\mathbb{Z}_6$	$\mathbb{Z}\times\mathbb{Z}_3\times\mathbb{Z}_6$
$p6$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_6\times\mathbb{Z}_3$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_6\times\mathbb{Z}_3$
$p6m$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_2^2$	$2\mathbb{Z}\times\mathbb{Z}_6\times\mathbb{Z}_2^3$

- Fermionic/bosonic root phases are denoted with red/blue color.
- No 1D block state decoration: no 1D topological insulator without symmetry or with a Z_2 onsite symmetry. (JH Zhang, S Yang, Y Qi, ZC Gu, Phys. Rev. Research 4, 033081 (2022)

Classification of TSC protected by point group symmetry in 3D

G_b	E_0^{2D}	E_0^{1D}	E_0^{0D}	$\mathcal{G}_0$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_{16}$
C_{2h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{2v}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
$D_{2h} = V_h$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C_{4h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
C_{4v}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
$D_{2d} = V_d$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
D_{4h}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_{16}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_{16}$
D_{3d}	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8$
C_{6h}	$\mathbb{Z}_8$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{6v}	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
D_{3h}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$
D_{6h}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
T_d	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
O_h	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$

G_b	$E_{1/2}^{\mathrm{2D}}$	$E_{1/2}^{\mathrm{1D}}$	$E_{1/2}^{\mathrm{0D}}$	$\mathcal{G}_{1/2}$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{2h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{2v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$D_{2h} = V_h$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C_{4h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{4v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$D_{2d} = V_d$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
D_{4h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_{3d}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{3h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{6h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{6v}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_{3h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
D_{6h}	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
T_d	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
O_h	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$

For usual fermion systems with spin-1/2, all 2D and 1D block states decorations are all trivial, there would be no interesting crystalline TSC.

Although spinless fermion systems are not quite natural, SC with co-planar magnetic order or applying strong magnetic field may still possible to realize them in experiments.

(Jian-Hao Zhang, Yang Qi, Zheng-Cheng Gu, arXiv:2204.13558 (2022))

Classification of TI protected by point group symmetry in 3D

G_b	$E^{2D}_{0,U(1)}$	$E^{1D}_{0,U(1)}$	$E^{0D}_{0,U(1)}$	$\mathcal{G}_{0,U(1)}$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{2h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{2v}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
$D_{2h} = V_h$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^7$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
C_{4h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{4v}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
$D_{2d} = V_d$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$
D_{4h}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^7$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
D_{3d}	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{6h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{6v}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$
D_{3h}	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
D_{6h}	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^7$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^3$
T_d	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
O_h	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^4$

G_b	$E^{2D}_{1/2,U(1)}$	$E^{1D}_{1/2,U(1)}$	$E^{0D}_{1/2,U(1)}$	$\mathcal{G}_{1/2,U(1)}$
C_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_i = S_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_2	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
$C_s = C_{1h}$	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{2h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
$D_2 = V$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_{2v}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$D_{2h} = V_h$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$
C_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_4	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
C_{4h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
D_4	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
C_{4v}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$D_{2d} = V_d$	$\mathbb{Z}_8 \times \mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8 \times \mathbb{Z}_2^5$
D_{4h}	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$
C_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
S_6	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$
D_3	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3v}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
D_{3d}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
C_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
C_{3h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
C_{6h}	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
D_6	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
C_{6v}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
D_{3h}	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^3$
D_{6h}	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^5$
T	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$
T_h	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^3$
T_d	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2^2$
O	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
O_h	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$	$\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2^4$

Summary and future works

- We classify and construct all fermionic SPT phases and crystalline topological superconductors/topological insulators in 2D interacting fermion systems.

- All scheme can be generalized into 3D, for both internal symmetry and space group symmetry

- Full classification for all 230 space group.

- Resources for measurement based quantum computation.

- Topological phase transitions for interacting topological superconductors.

- Experimental realization for interacting crystalline topological insulators and superconductors.