Topological Quantum Computing

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August 12, 2005

Summary

- Why?
- What?
- How?

Why?

Standard Quantum Computation Model

- Qubits are encoded into a superposition of orthogonal states of a two level system: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

- Quantum computer is a collection of n qubits.

- Calculations are performed through the action of a universal set of m quantum gates $\{U_1, U_2, \ldots, U_m\}$ on one or more qubits.

- Calculation result is read by projecting end state onto $\{|0\rangle, |1\rangle\}$ basis.

Decoherence and Error Correction

- In real world quantum computer will fail due to decoherence effects

$$|\psi\rangle = a|0\rangle + b|1\rangle \rightarrow |\psi\rangle = |0\rangle \text{ or } |1\rangle$$

- . . . unless we protect the qubits!

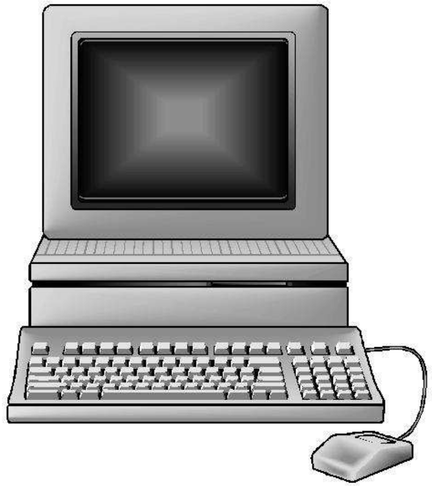
- Shor's and Steane's error correction codes in 1995 allowed clever encoding of qubits and gates

- In principle, using error correction, fault-tolerant computation can be performed

- However, to give reliable results, large hierarchy of error correction mechanism are needed.

- Need better hardware. . .

Imperfect Hardware



Reliable Hardware

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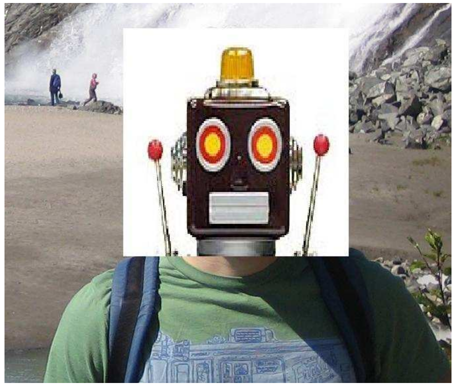

Imperfect Hardware

Reliable Hardware

Topology

- In math: Topology is study of topological spaces

- Topological space is composed of set X and a collection of subsets T such that:

  1. Union of any collection of subsets of T is still in T
  2. Intersection of any pair of subsets of T is still in T
  3. T contains the empty set and X

- Examples of topological spaces: real numbers

Topological properties of an object are those which are unchanged by smooth deformations

$\Leftrightarrow$

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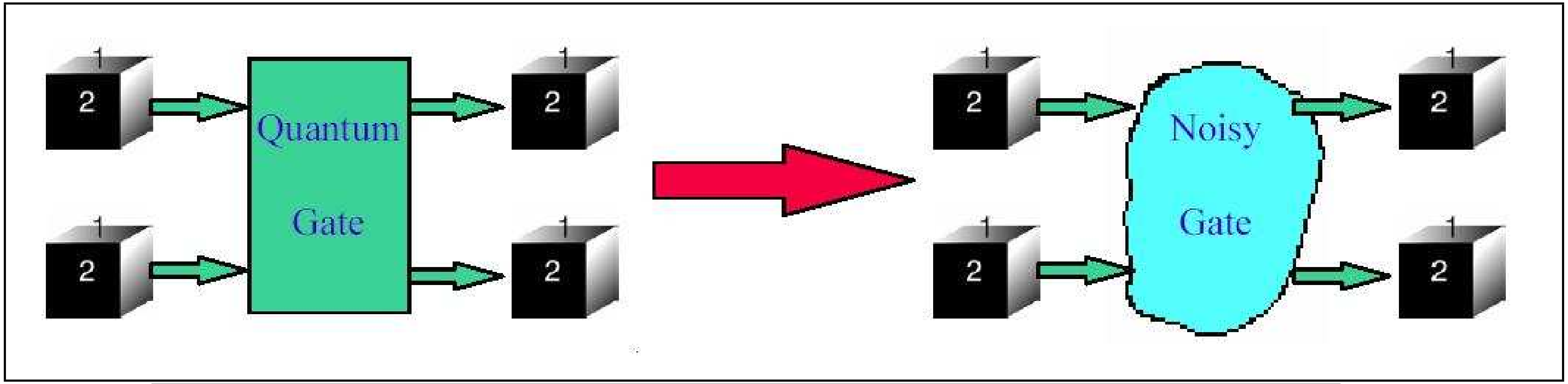
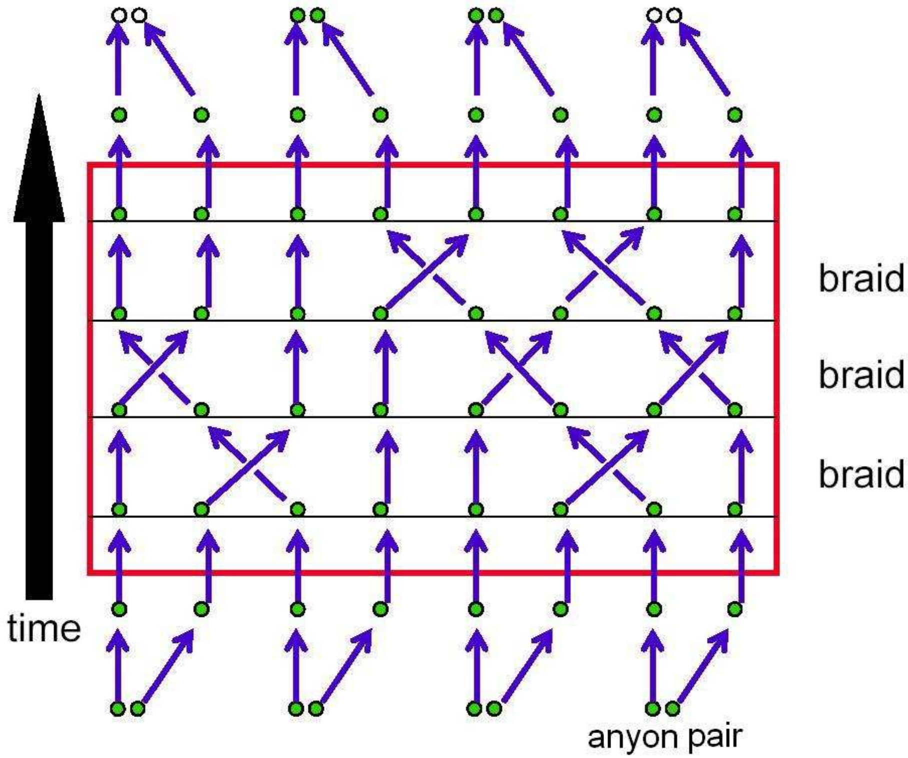


We'd like to use this intrinsic fault tolerance to our advantage

Topological Quantum Computer



What?

Indistinguishable particles

- Suppose I have two identical particles at x_1 and x_2 described by $|\psi\rangle = |x_1 x_2\rangle$
- Permutation operator, P, switches particles 1 and 2:

$$\begin{aligned} P|x_1 x_2\rangle &= e^{i\phi}|x_2 x_1\rangle \\ P^2|x_1 x_2\rangle &= e^{2i\phi}|x_1 x_2\rangle \end{aligned}$$

- However, since $P^2 \equiv I$, we need $e^{2i\phi} = 1$. This implies $e^{i\phi} = \pm 1$.
- Fermions ($\phi = 2\pi \cdot \frac{1}{2}$) and bosons ($\phi = 2\pi \cdot 1$)

In all dimensions of space, particles are either bosons or fermions. But there is an exception. . . ...

Anyons

- Fermions

$$\Psi(x_1, x_2) = -\Psi(x_2, x_1) = \Psi(x_2, x_1)e^{i\pi}$$

- Bosons

$$\Psi(x_1, x_2) = \Psi(x_2, x_1) = \Psi(x_2, x_1)e^{i2\pi}$$

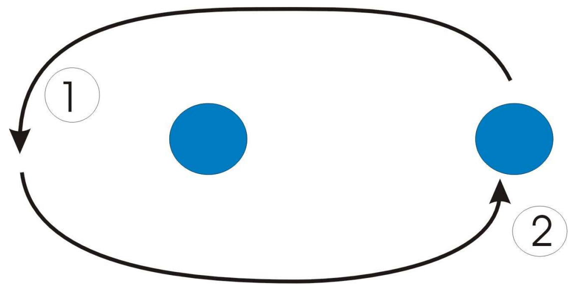
- Anyons

$$\Psi(x_1, x_2) = \Psi(x_2, x_1) = \Psi(x_2, x_1)e^{i\pi(1+m)}$$

anyons have fractional statistics [1]

$$0 \leq m \leq 1$$

- Swapping particles is represented as rotation. Double swap is a full loop.



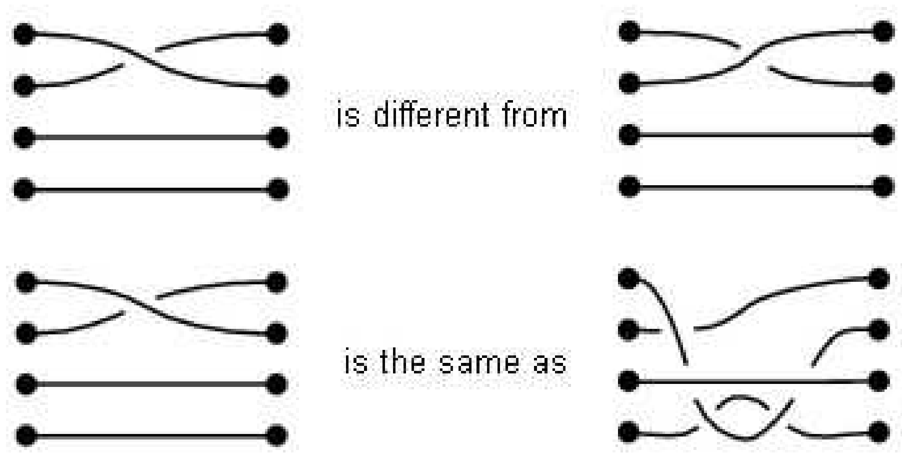
- In 3D, the particle's path can be smoothly contracted to the trivial one.

- In 2D, you can't do this: rotation by 2π eigenvalues are not limited to ± 1

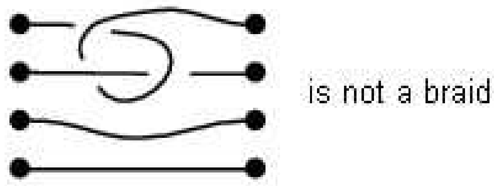
- Anyons only exist in 2D due to different topological properties of the rotation group SO(2).

Braid Group

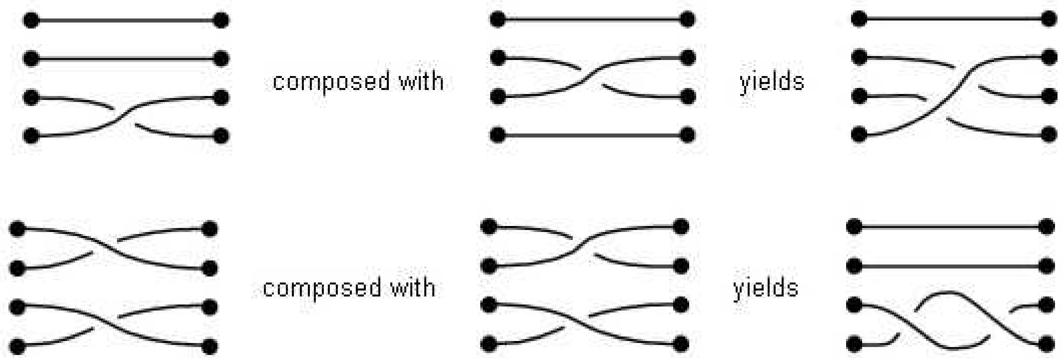
Braid group B_n consists of different ways in which n particles can be braided.



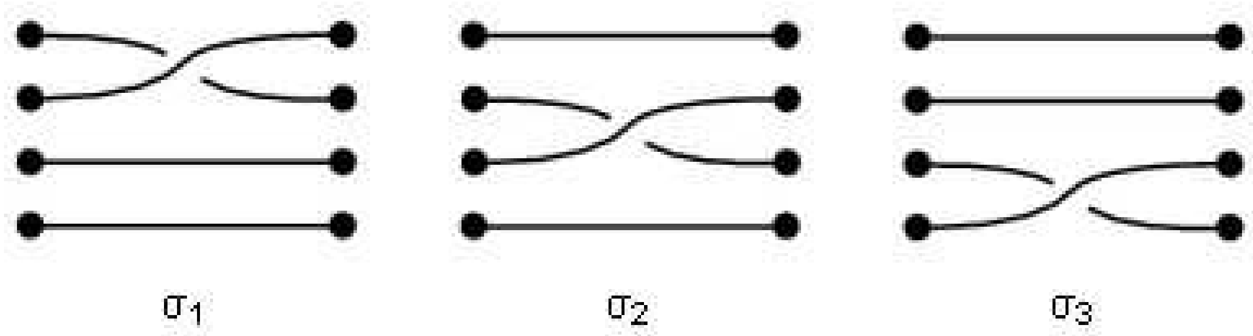
- Knots are not allowed



- Braids can be composed



- Any braid can be constructed from exchanges of neighboring particles. These exchanges are the generators for the group. There are 3 generators for the B_4 braid group:



- Braid group is infinite and thus has an infinite number of representations. It has 1D, as well as higher dimension representations.

- Identical particles that transform as a 1D representation of the braid group are *abelian anyons*: generators are then represented as phase shifts:

$$\sigma_j = e^{i\phi_j}$$

- Braid group has nonabelian representations. For example, generators can be represented as non-commuting matrices instead of phase shifts.

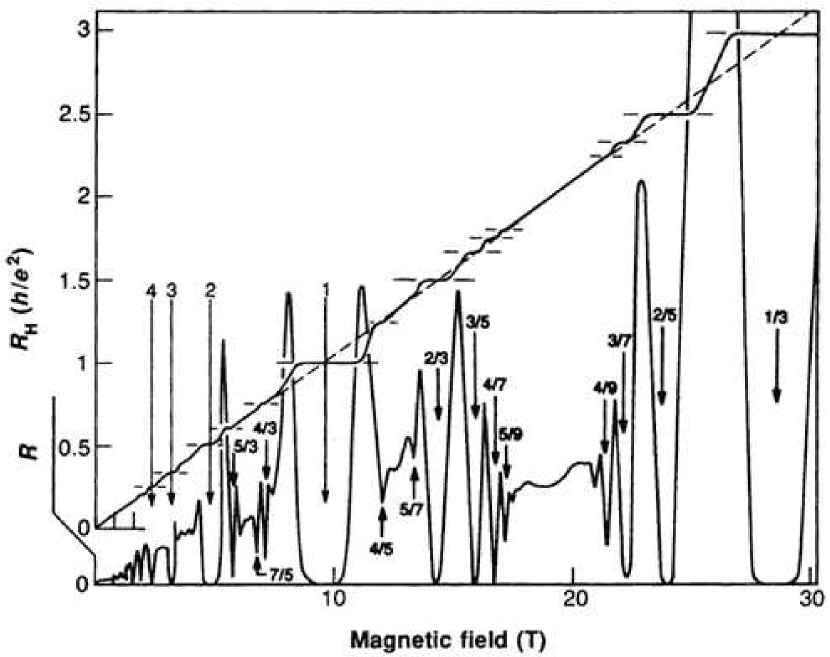
- Identical particles that transform as such are *nonabelian anyons*.

- Irreducible representation of B_n from n anyons acts on a topological vector space V_n. Dimension of V_n increases exponentially with n.

- Depending on the type of nonabelian anyon, image of representation may be dense in SU(D_n).

Universal quantum computation is possible with braiding of nonabelian anyons!

Science Fiction? No!



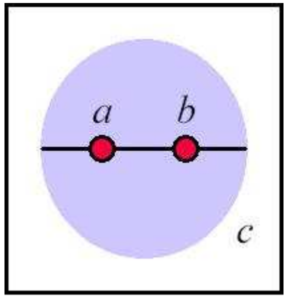
Fractional Quantum Hall Effect

- Many 2D systems exist in nature such as 2D electron gases and rotating Bose gases
- In either of these systems, there are abelian and nonabelian quasiparticles.
- Lots of weird stuff, like fractional charge.

A Simple Nonabelian Anyon Model

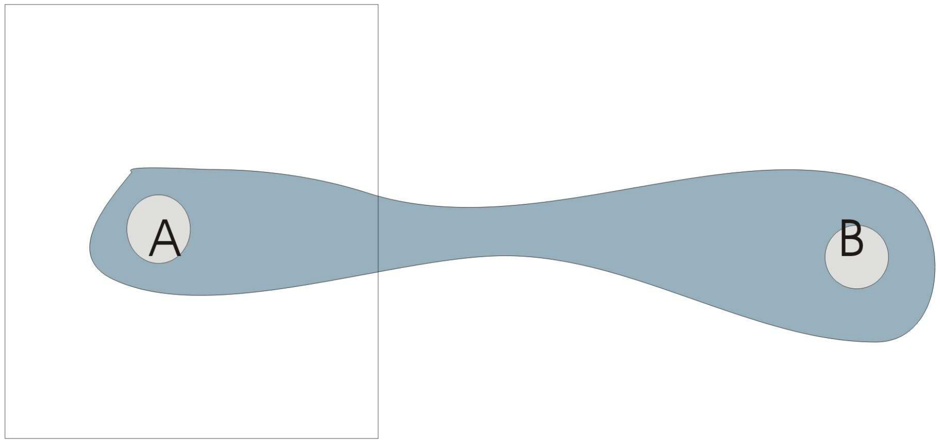
- Suppose we have a finite list of particle labels $\{a, b, c, \ldots\}$ indicating value of a conserved quantity that a particle can carry (like charge).

- Fusion rules are expressed as $a \times b = \sum_c N_{ab}^c c$. The N_{ab}^c distinct ways in which $\{a, b\} \to c$ form an orthonormal basis set of a Hilbert space V_{ab}^c called *fusion space*.



- If, for at least one pair of labels ab, $\sum_c N_{ab}^c \geq 2$, the anyon model is nonabelian.

- Topological Hilbert space! (Information about c isn't localized)



- Intrinsical robustness against decoherence.

- We can use this Hilbert space to encode quantum information.

Example: Fibonacci Anyons

- Charges can take two different values: 0 and 1. All anyons have charge 1.
- Simple fusion rule: $\bullet \times \bullet = 0 + 1$

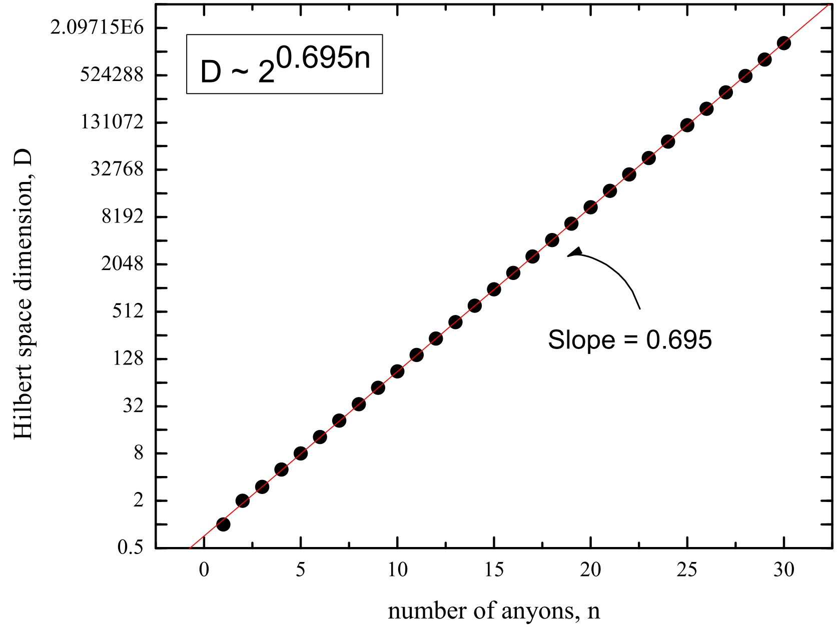
$$|\bullet\rangle + |\bullet\rangle = |1\rangle \text{ or } |0\rangle$$

- This describes nonabelian anyons because fusion gives 2 distinct values.
- Called Fibonacci anyons because n anyons span a Hilbert space of dimension equal to the $n + 1$ Fibonacci number.

- Example: 3 anyons → 3 states:

$$|\,(\bullet,\bullet)_0\,\bullet\rangle_1$$
$$|\,(\bullet,\bullet)_1\,\bullet\rangle_1$$
$$|\,(\bullet,\bullet)_1\,\bullet\rangle_0$$

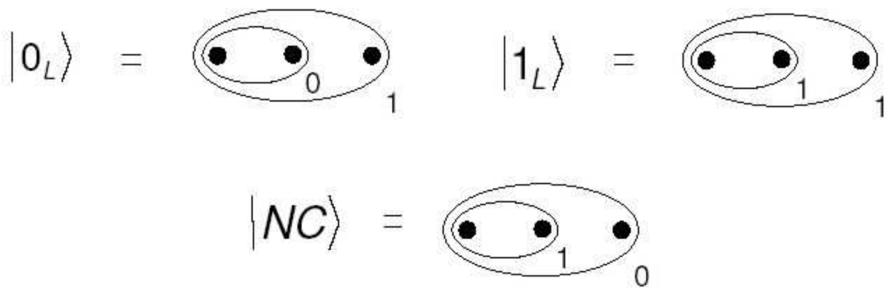
- Asymptotically, 0.694 qubits encoded by each anyon. Non locality!



How?

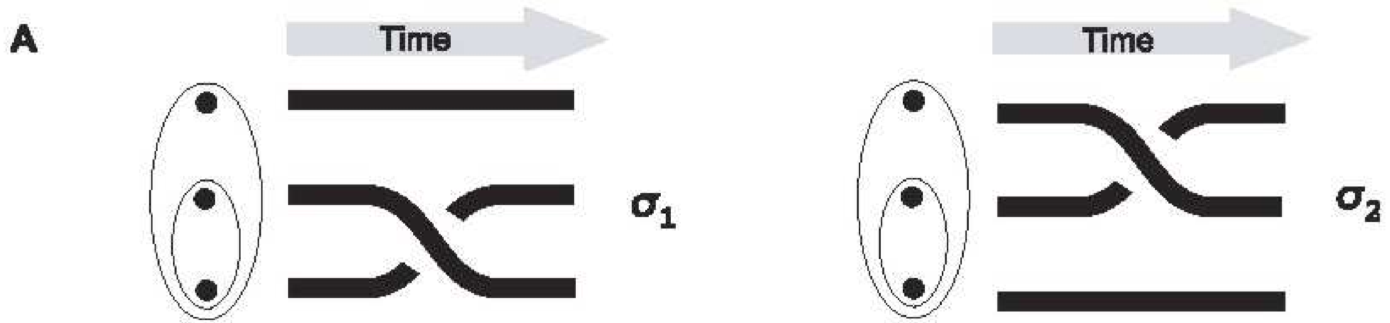
Using Fibonacci Anyons to Implement Quantum Gates

- Bonesteel's group showed nice implementation of quantum gates with Fibonacci anyons: quant-ph/0505065



- Logical qubits are encoded with 3 anyons.

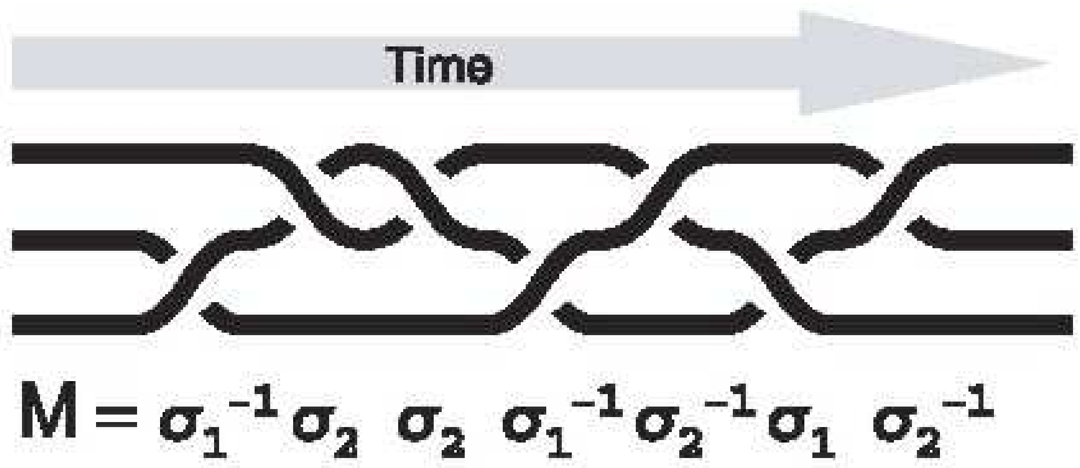
- Matrices σ_1 and σ_2 are braid generators acting on Hilbert space produced by 3 anyons in qubits



$$\sigma_1 = \left[\begin{array}{cc|c} e^{-i4\pi/5} & 0 & 0 \\ 0 & -e^{-i2\pi/5} & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right] \qquad \sigma_2 = \left[\begin{array}{cc|c} -\tau e^{-i\pi/5} & -i\sqrt{\tau}\, e^{-i\pi/10} & 0 \\ -i\sqrt{\tau}\, e^{-i\pi/10} & -\tau & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right]$$

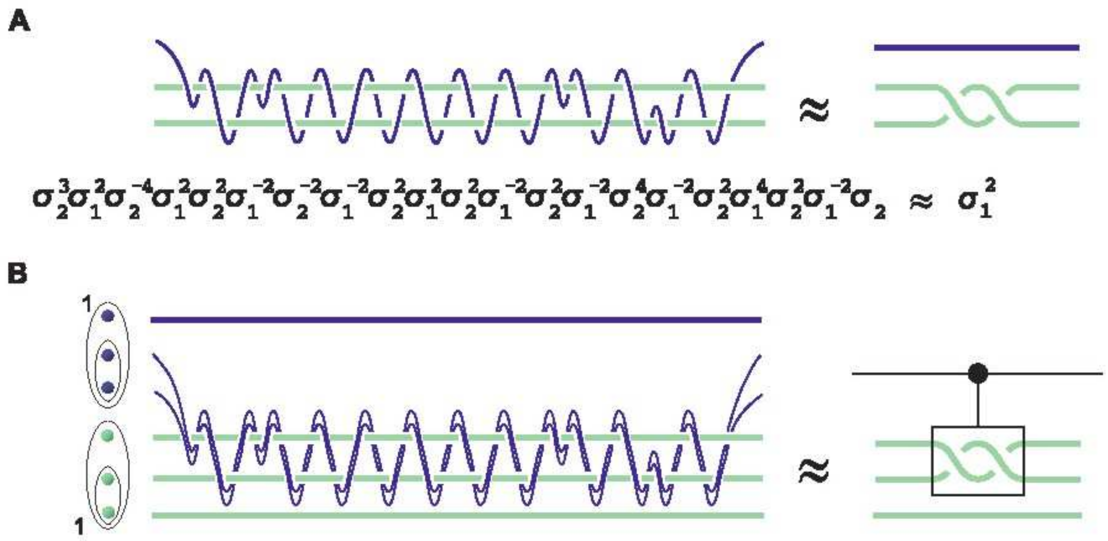
- Only upper 2x2 block acts on computational basis (where total charge=1)

- General 3-qubit braid by successively applying σ_1, σ_2 and their inverses



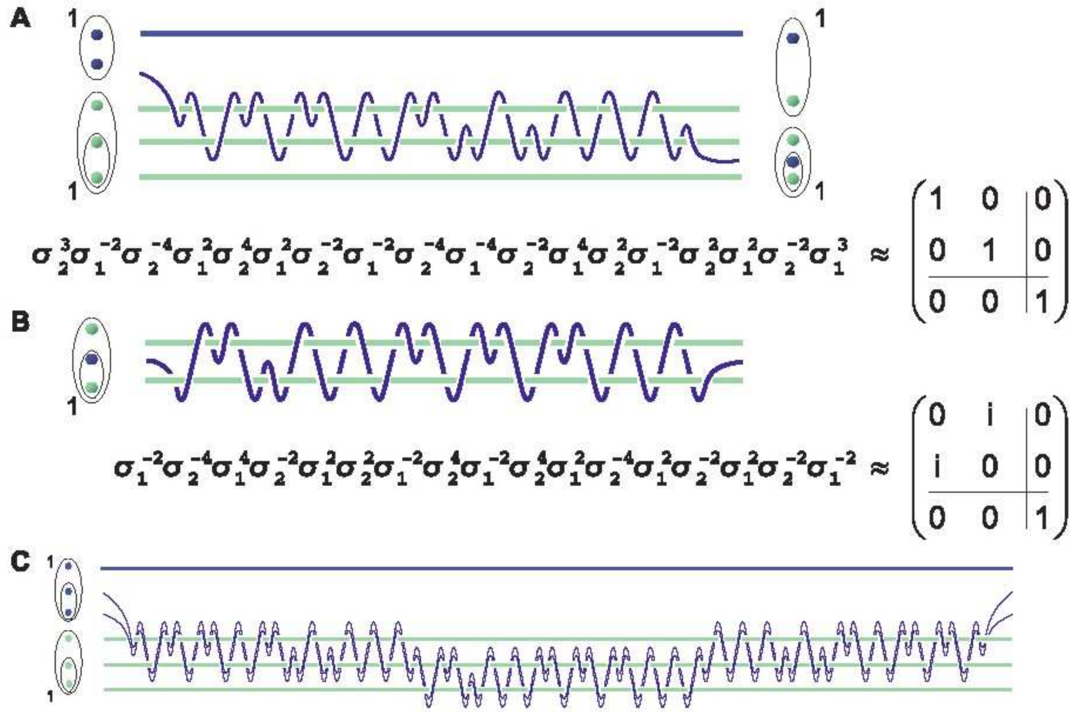
- Through brute force search, any single qubit gate can be approximated

- Controlled rotation gate. Resulting 2-qubit gate is a controlled rotation of target qubit.



- Together with single qubit gates, universal quantum computing.

CNOT


A
1
1
1
1
$\sigma_2^{3}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{-4}\sigma_2^{-2}\sigma_1^{4}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{3} \approx \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$
B
1
$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} \approx \left(\begin{array}{cc|c} 0 & i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$
C
1
1

Conlusions

- Topological Quantum Computing is desirable because of intrinsic decoherence resistance
- To do TQC, need nonabelian anyons.
- Fortunately, they DO exist
- TQC could be carried out by braiding these nonabelian anyons

What I'm doing

- Surprisingly, nobody has ever measure this topological phase
- I'm looking at rotating bosons and trying to find nonabelian states

Thanks for listening!