Purification of *large* multi-party states

An analytical approach

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August 2006
Canadian Quantum Information Students' Conference

Outline

1. Introduction
   - What, How and Why
   - Technical prerequisites
2. Base protocols
   - Post-selection
   - Error correction
3. Band-aid protocols
   - The protocols
   - Performance

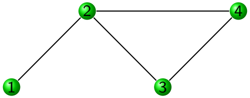
What, How and Why

- What
  - Create as pure a state as possible with imperfect operations
  - Use minimum resources (unpurified entangled states)
  - Analytical estimates of performance/thresholds
- How
  - Use multiple copies and/or band-aids
  - Locality properties $\Rightarrow$ Decoupled recursion relations
- Why
  - Improve thresholds for fault tolerant one way QC
  - Create long range entanglement e.g. Quantum Repeaters
  - Very few analytical results on purification

Error

Information scientist

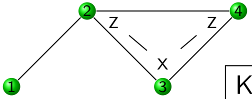
Graph States
Creation and properties



- Vertex $\equiv$ qubit prepared in the $|+\rangle$ state
- Edge $\equiv$ a Control-Phase gate

Graph States

Creation and properties



$$\boxed{K_3 = X_3\, Z_2\, Z_4}$$

- Vertex $\equiv$ qubit prepared in the $|+\rangle$ state
- Edge $\equiv$ a Control-Phase gate
- Graph basis $\{|\vec{\mu}\rangle\}$ (of $\mathfrak{C}^{2^{\otimes N}}$)

$$K_j\,|\vec{\mu}\rangle = (-1)^{\mu_j}\,|\vec{\mu}\rangle\,; \quad \vec{\mu} \in \{0,1\}^N$$

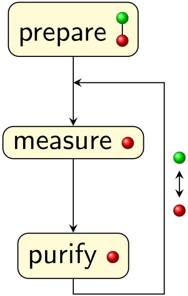
$$K_j = X_j \bigotimes_{i \in \mathrm{neigh}(j)} Z_i$$

- Ideal preparation gives $|\vec{\mu} = \vec{0}\rangle$
- Pauli noise gives a mixture of $\{|\vec{\mu}\rangle\}$

The Purification Schema

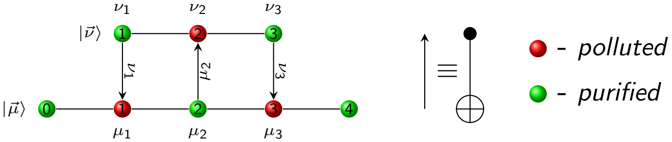
Works on bi-colorable graph states only

- Prepare a noisy bi-colorable graph state
- Measure K_j for qubits of one color
- Use the measurement results to purify those qubits
  - Post-selection
  - Error correction
- Side-effect: qubits of the other color are polluted
- Concatenate and repeat for qubits of the other color
- Run repeatedly to reach fixed point



Measuring K_j

The basic step in all purification protocols

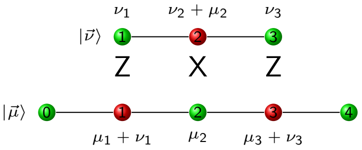


- The MCNOT

$$\mu_j \mapsto \begin{cases} \mu_j + \nu_j & j \in \textcolor{red}{\bullet} \\ \mu_j & j \in \textcolor{green}{\bullet} \end{cases}$$

Measuring K_j

The basic step in all purification protocols



$\color{red}\bullet$ - *polluted*

$\color{green}\bullet$ - *purified*

- The MCNOT

$$\mu_j \mapsto \begin{cases} \mu_j + \nu_j & j \in {\color{red}\bullet} \\ \mu_j & j \in {\color{green}\bullet} \end{cases}$$

- Measure $K_2 = X_2 Z_1 Z_3$

$$1 \equiv 0 \qquad\qquad -1 \equiv 1$$

The Variables

How we analyze purification protocols

- Arbitrary stabilizer element

$$\mathsf{K}_{\vec{a},\vec{b}} := \prod_{i=1}^{N_{\bullet}} \mathsf{K}_i^{a_i} \times \prod_{j=1}^{N_{\bullet}} \mathsf{K}_j^{b_j}$$

- Diagonal density matrices

$$\rho = \frac{1}{2^{N_{\bullet}+N_{\bullet}}} \sum_{\vec{a},\vec{b}} \langle \mathsf{K}_{\vec{a},\vec{b}} \rangle \, \mathsf{K}_{\vec{a},\vec{b}}$$

- Recursion Relations

$$\langle \mathsf{K}_{\vec{a},\vec{b}} \rangle' = f(\{\langle \mathsf{K}_{\vec{c}} \rangle\}); \quad |\vec{c}| \leq |\vec{a}| + |\vec{b}|$$

Modeling Noise

Noisy gate $\equiv$ ideal gate + depolarizing channel

- Noisy gates: Ideal gate + Depolarizing channel
- Single qubit

$$T_a(p_1) = (1 - p_1)[\mathrm{I}_a] + \frac{p_1}{3}([\mathrm{X}_a] + [\mathrm{Y}_a] + [\mathrm{Z}_a])$$

- Two qubits

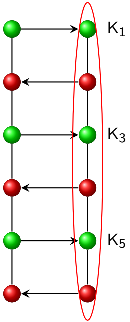
$$T_{a,b}(p_2) = (1 - p_2)[\mathrm{I}_{a,b}] + \frac{p_2}{15}([\mathrm{I}_a \otimes \mathrm{X}_b] + \cdots + [\mathrm{Z}_a \otimes \mathrm{Z}_b])$$

- The various types of noise
  - Creation error
  - Gate error
  - Measurement error
  - Storage error

Post-selection Protocol†

The most effective, but inefficient and difficult to analyze

- Two identical copies at each round
- Accept iff all measurements succeed
- Removes errors to lowest order



† Aschauer, Briegel and Dür [03]

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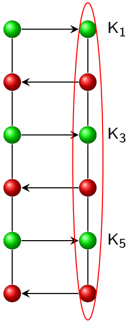

Advantages	Disadvantages
High threshold	Cannot be used for large states
Fixed point less sensitive to noise	Difficult to analyze

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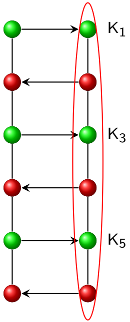

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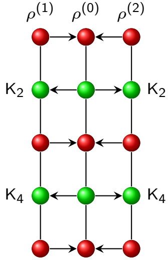
3-Copy Protocol†

The obvious modification



$$\mu^{(1)}_{\color{green}\bullet} \mapsto \mu^{(1)}_{\color{green}\bullet} + \mu^{(0)}_{\color{green}\bullet}$$
$$\mu^{(0)}_{\color{red}\bullet} \mapsto \mu^{(0)}_{\color{red}\bullet} + \mu^{(1)}_{\color{red}\bullet} + \mu^{(2)}_{\color{red}\bullet}$$

- Measure $K_{\color{green}\bullet}$ on $\rho^{(1)}$ and $\rho^{(2)}$
- Flip $K_{\color{green}\bullet}$ on $\rho^{(0)}$ iff measurements are both one on $\rho^{(1)}$ and $\rho^{(2)}$
- Error correction at j^{th} qubit fails iff $\mu_j = 1$ on at least two states

$$\langle K_{\color{green}\bullet} \rangle \mapsto \frac{1}{2}\left(3 - \langle K_{\color{green}\bullet} \rangle^2\right) \langle K_{\color{green}\bullet} \rangle$$

- Double the pollution

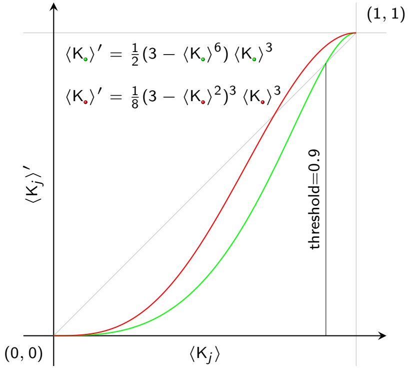
$$\langle K_{\color{red}\bullet} \rangle \mapsto \langle K_{\color{red}\bullet} \rangle^3$$

$$\langle K_{\vec{a},\vec{b}} \rangle' = \frac{1}{2^{|\vec{a}|}} \sum_{\vec{a}_1,\vec{a}_2 \ll \vec{a}} (-1)^{\vec{a}_1 \cdot \vec{a}_2} \langle K_{\vec{a}+\vec{a}_1+\vec{a}_2,\vec{b}} \rangle \langle K_{\vec{a}_1,\vec{b}} \rangle \langle K_{\vec{a}_2,\vec{b}} \rangle$$

† Goyal, Raussendorf, McCauley [06]

3-Copy Protocol

Recursion relations (noiseless gates)



- Concatenate k times

$$\frac{1 - \langle K_j(k) \rangle}{2} =: P_j(k) \leq \left(\frac{P_j(0)}{P_{th}} \right)^{2^k}$$

- Needs 3^{2k} states
- Noise multiplies each term by

$$0 < f(1 - p, \vec{a_1}, \vec{a_2}, \vec{b}) < 1$$

$$\langle K_{\vec{a},\vec{b}} \rangle' = \frac{1}{2^{|\vec{a}|}} \sum_{\vec{a}_1, \vec{a}_2 \ll \vec{a}} (-1)^{\vec{a}_1 \cdot \vec{a}_2} \langle K_{\vec{a}+\vec{a}_1+\vec{a}_2, \vec{b}} \rangle \langle K_{\vec{a}_1, \vec{b}} \rangle \langle K_{\vec{a}_2, \vec{b}} \rangle$$

Can we be more intelligent?

Combining post-selection and error correction

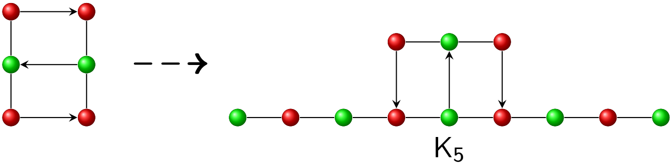
Post-selection	Error correction
High threshold	Local
FP $\sim 1 - p(d+1)$	Analyzable

Can we be more intelligent?

Combining post-selection and error correction

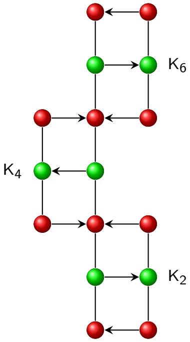
Post-selection	**Error correction**
High threshold	Local
FP $\sim 1 - p(d+1)$	Analyzable

- The Band-aid
  - Small post-selected GHZ state $(d+1)$ qubits
  - Use to measure each K_j individually
  - Trust band-aid since it has been pre-purified
    - Ideal gates $\Rightarrow$ purity transfer



The Band-aid Protocol†

Making post-selection local



- Post-select the band-aids
- Use them to purify $K_{\bullet}$
- Repeat $\bullet \leftrightarrow \bullet$
- Gate and measurement errors (rate p)

$$\langle K_{\bullet}\rangle' = (1-p)^{\frac{d(d+7)+4}{2}}\beta^{d+1}$$

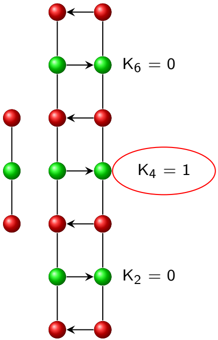
- Fixed point behavior

$$\langle K_j\rangle \to 1 - \frac{d(3d+11)+6}{2}p + O(p^2)$$

† Goyal, McCauley, Raussendorf [06]

The Conditional Band-aid Protocol†

Best of the breed?



- Identical states for first measurement
- Ambiguous syndrome $\Rightarrow$ second measurement
- Repeat 🟢 $\leftrightarrow$ 🔴
- Concatenate
- Fixed point behavior

$$\langle K_j \rangle \rightarrow 1 - 2(d+1)p + O(p^2)$$

$$\langle K_{\color{red}\bullet} \rangle' \geq \alpha \left(1 - \frac{d}{2}(1-\gamma^d)(1 - \langle K_{\color{green}\bullet} \rangle)\right) \langle K_{\color{red}\bullet} \rangle^2$$

$$\alpha = (1-p)^{d+1} \qquad \gamma = (1-p)^2 \beta$$

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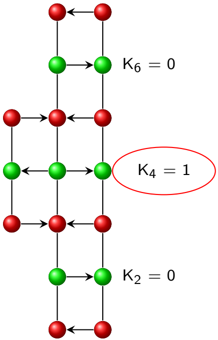

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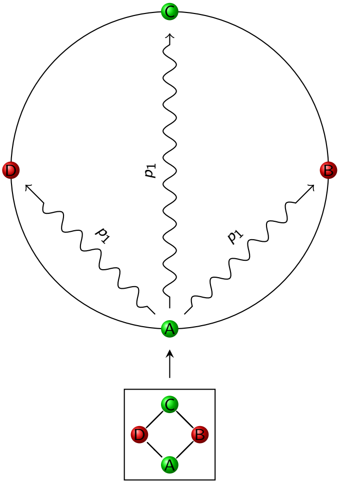
Trade-off Curves

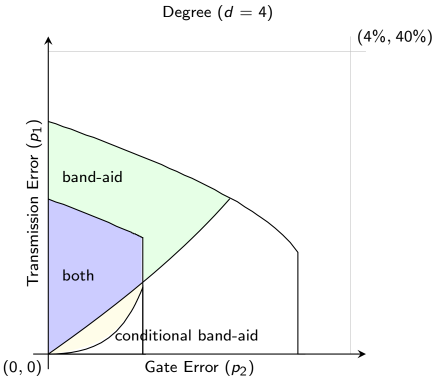
With gate, storage and measurement errors



- No purification
  - Prepared locally
  - Each qubit sent/stored over a noisy quantum channel
- Purification
  - Band-aids and large state prepared locally
  - Both distributed over noisy channels
  - Band-aids purified by post-selection (LOCC)
  - Band-aids used to purify large state (LOCC)

Trade-off Curves

With gate, storage and measurement errors



$$\langle K_j \rangle_{\text{initial}} = (1 - p_2)^{\frac{d(d+1)}{2}} (1 - p_1)^{d+1}$$

- No purification
  - Prepared locally
  - Each qubit sent/stored over a noisy quantum channel
- Purification
  - Band-aids and large state prepared locally
  - Both distributed over noisy channels
  - Band-aids purified by post-selection (LOCC)
  - Band-aids used to purify large state (LOCC)

Protocol Roundup

- Performance is independent of size
- Analyzable even with noisy purification operations
- Can tolerate 3% gate or 30% local error

	Threshold	**Fixed Point**	**Locality**
3 Copy	Poor	Stable	Local
Band-aid	Excellent	Sensitive	Local
C. B.	Good	Stable	Weakly Non-local

Summary

- Efficient purification of large states is possible with imperfect operations
- Analytical analysis of purification protocols is possible by taking advantage of locality
- Outlook
    - Classify protocol families: Threshold, Fixed-point purity, Resource usage
    - Analyze noise structure at fixed point
    - Make connection to error correction using CSS codes

Noise Structure at the Fixed Point (3-Copy)

A bonus of this method of analysis

- $\langle \mathsf{K}_{\vec{a},\vec{b}} \rangle' = f(\{\langle \mathsf{K}_{\vec{c}} \rangle\}); \quad |\vec{c}| \leq |\vec{a}| + |\vec{b}|$
  - $x' = px^3 + qx + r$
  - $|\vec{a}| + |\vec{b}| \leq 2 \Rightarrow$ unique fixed point
- Recursion relations preserve the property $\langle \mathsf{K}_i \, \mathsf{K}_j \rangle = \langle \mathsf{K}_i \rangle \langle \mathsf{K}_j \rangle$ iff $\text{neigh}(i) \cap \text{neigh}(j) = \emptyset$
- Analyze purification of ideal state
  - No two point correlations at fixed point!
- Regard the 3-copy protocol as a technique for creating states with only local errors

Generalization to Arbitrary Density Matrices

- The Depolarizing Operator

$$\mathcal{D} := \left(\prod_{\vec{a} \in \{0,1\}^{N_\bullet}} \frac{[I] + [\mathrm{K}_{\vec{a},\vec{0}}]}{2} \right) \left(\prod_{\vec{b} \in \{0,1\}^{N_\bullet}} \frac{[I] + [\mathrm{K}_{\vec{0},\vec{b}}]}{2} \right)$$

- $\mathcal{D}\rho \equiv \rho_D$ is diagonal in the graph basis
- $\mathcal{D}$ commutes with the protocol operations
- A recursion relation valid for ρ_D will also be valid for ρ

$$\langle \mathrm{R}(\rho) \rangle_{\vec{a},\vec{b}} = \langle \mathcal{D} \circ \mathrm{R}(\rho) \rangle_{\vec{a},\vec{b}} = \langle \mathrm{R} \circ \mathcal{D}(\rho) \rangle_{\vec{a},\vec{b}} = \langle \mathrm{R}(\rho_D) \rangle_{\vec{a},\vec{b}}$$